

C A L C U L U S

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Preface

Calculus is at once the most important and the most difficult subject encountered early by students of mathematics; introductory courses often succeed only in turning students away from mathematics, and from the many subjects in which the calculus plays a major role.

The present text introduces calculus in the informal manner adopted in my *Arithmetic* [1], a manner endorsed by Lakatos [2], and by the following words of Lanczos from his preface to [3]:

Furthermore, the author has the notion that mathematical formulas have their "secret life" behind their Golem-like appearance. To bring out the "secret life" of mathematical relations by an occasional narrative digression does not appear to him a profanation of the sacred rituals of formal analysis but merely an attempt to a more integrated way of understanding. The reader who has to struggle through a maze of "lemmas", "corollaries", and "theorems," can easily get lost in formalistic details, to the detriment of the essential elements of the results obtained. By keeping his mind on the principal points he gains in depth, although he may lose in details. The loss is not serious, however, since any reader equipped with the elementary tools of algebra and calculus can easily interpolate the missing details. It is a well-known experience that the only truly enjoyable and profitable way of studying mathematics is the method of "filling-in details" by one's own efforts.

The scope is broader than is usual in an introduction, embracing not only the *differential* and *integral* calculus, but also the *difference* calculus so useful in approximations, and the *partial derivatives* and the *fractional* calculus usually met only in advanced courses. Such breadth is achievable in small compass not only because of the adoption of informality, but also because of the executable notation employed. In particular, the array character of the notation makes possible an elementary treatment of partial derivatives in the manner used in tensor analysis.

The text is paced for a reader familiar with polynomials, matrix products, linear functions, and other notions of elementary algebra; nevertheless, full definitions of such matters are also provided.

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Appendix A: Language Summary

Chapter 1: INTRODUCTION

A. NOTATION AND TERMINOLOGY

The notation (J Version 6.2) used exclusively in this text has the following characteristics:

1. It is completely defined in a thirty-five-page dictionary [4].
2. It is directly executable on a wide range of computers [5].
3. It applies directly to vectors, matrices, and higher-rank arrays.
4. It incorporates *operators* (in the sense of Heaviside) in a systematic way that makes them accessible at an elementary level. For example, the symbol $\int$ denotes an operator that applies to a function to produce another function, as in $\int +$ and $\int *$ to replace the conventional sigma and pi notation for sums and products. Similarly, $\&$ applies to two arguments, *bonding* them to define a new function, as in $\&3$ for the *triple* function, and in $\wedge\&3$ for the *cube* function, and $10\&.$ for the *base-10* logarithm.
5. The terminology used is drawn more from normal English than from mathematics. In particular:
 - a) Functions such as $+$ and $*$ and $\wedge$ are also referred to as *verbs* (because they *act* upon *nouns* such as 3 and 4), and operators such as $\int$ and $\&$ are accordingly called *adverbs* and *conjunctions*, respectively.
 - b) The symbol $=.$ used in assigning a name to a referent is called a *copula*, and the names *credits* and *sum* used in the sentences *credits* $=.$ 24.5 17 38 and *sum* $=.$ $\int +$ are referred to as *pronouns* and *proverbs*, respectively.
 - c) Vectors and matrices are also referred to by the more suggestive terms *lists* and *tables*.

Because the notation is executable, the computer can be used to explore and elucidate topics with a clarity that can only be appreciated from direct experience of its use. The reader is therefore urged to use the computer to do the exercises provided for each section, as well as other experiments that may suggest themselves.

To avoid distractions from the central topic of the calculus, we will introduce the necessary notation with a minimum of comment, assuming that the reader can grasp the meaning of new notation from context, from simple experiments on the computer, from the lan-

guage summary in Appendix A, from the *Introduction and Dictionary* [4], or from the study of more elementary texts such as *Arithmetic* [1]. The remainder of this section is an annotated computer dialog that introduces the main characteristics of the notation:

3.45+6.78+0.01	Plus
10.24	
2*3	Times
6	
2^3	Power
8	
1 2 3 * 4 5 6	Lists or vectors
4 10 18	
2 < 3 2 1	<i>Less than</i>
1 0 0	
2 <. 3 2 1	<i>Lesser of</i> (Minimum). Related spellings denote related verbs
2 2 1	
(+: , -: , *: , %:)	16 <i>double, halve, square, square root</i>
32 8 256 4	
+/4 5 6	<i>/ is the adverb insert</i>
15	
4+5+6	
15	
*/4 5 6	
120	
3-5	Verbs are ambivalent, with a meaning that depends on context; the symbol - denotes subtraction or negation according to context.
_2	
-5	
_5	
2^1 2 3	
2 4 8	
^1 2 3	The exponential function
2.71828 7.38906 20.0855	
*/4 5 6	A <i>derived</i> verb produced by an adverb is also ambivalent; the dyadic case of */ produces a multiplication table
120	
1 2 3 */ 4 5 6	
4 5 6	
8 10 12	
12 15 18	

```

3 -@^ 2 The conjunction @ applies its monad left argument
_9      (negation) atop its dyad right argument (power)

3 -&^ 2 The conjunction & applies its dyad left argument to
12.6965 the results of its monad right argument

a=. 1 2 3 The copula (=.) can be used to assign names
b=. 4 5 6 7 to nouns, verbs, adverbs, and conjunctions

powertable=. ^/
c=. a powertable b
c

1 1 1 1
16 32 64 128
81 243 729 2187

+/ c Adds together items (rows) of the table c
98 276 794 2316
+/"1 c The rank conjunction " applies its argument (in this
4 240 3240 case the function +/) to each rank-1 cell of c

x=. 4
x"0 a Applied to a noun, the rank conjunction
4 4 4 produces a constant function

from=. -- The adverb ~ commutes (reverses the
3 from 10 sense of) its verb argument

7
^~ 3 4 The monadic case replicates the argument
27 256

1+x*(3+x*(3+x*(1))) Parentheses provide punctuation as in
125 high-school algebra. However, there
1+x*3+x*3+x*1 is no precedence or hierarchy among
125 verbs; each applies to the result of the
(3*4)+5 entire phrase to its right
17
3*4+5
27

(+/ % #) b An isolated sequence of three verbs is a fork,
5.5 whose monadic case is defined as in this
(+/b) % (#b) example. The symbols % and # denote
5.5 divide and number of items, respectively

mean=. +/ % #
mean b
5.5

```

a (+ * -) d=. 4 5 6	The dyadic case of the fork
_15 _21 _27 (a+d) * (a-d)	
_15 _21 _27	
tithe=. %&10	The conjunction & (applied earlier
tithe 35	to verbs) also applies to a noun to
3.5	<i>bond</i> it to a dyad. The result is a
log=. 10&^.	corresponding monad
log 10 20 100	
1 1.30103 2	Sine (radian arguments)
sin=. 1&o.	Arguments 0, 1, and one-half pi
sin 0 1 1r2p1	
0 0.841471 1	
power=. ^&	A conjunction together with one of its
3 power x=. 1 2 3 4	arguments defines a corresponding
1 8 27 64	adverb
adv=. (&3) (@+:)	A sequence of adverbs defines an
^ adv 1 2 3 4	adverb
8 64 216 512	

We will write informal proofs by writing a sequence of sentences to imply that each is equivalent to its predecessor, and that the last is therefore equivalent to the first. For example, to show that the sum of the first n odd numbers is the square of n , we begin with:

```

]odds=. 1+2*i.n=. 8
1 3 5 7 9 11 13 15
|.odds
15 13 11 9 7 5 3 1
odds + |.odds
16 16 16 16 16 16 16 16
n#n
8 8 8 8 8 8 8 8

```

and then write the following sequence of equivalent sentences:

```

+/odds
+/.odds
-: (+/odds) + (+/.odds)
-: +/ (odds+|.odds)
+/-: (odds+|.odds)
+/n#n
n*n
*:n

```

EXERCISES

Solutions or hints will appear in bold brackets.
Make serious attempts before consulting them.

A1 To gain familiarity with the keyboard and the use of the computer, enter each of the sentences of Section A and verify that they produce the results shown in the text. Do not enter any of the comments that appear to the right of the sentences.

A2 To test your understanding of the notions illustrated by the sentences of Section A, enter variants of them, but try to predict the results before pressing the *enter* key.

A3 Enter `p=. 2 3 5 7 11` and predict the results of `+/p` and `*/p`; then review the discussion of parentheses and predict the results of `-/p` and `%p`.

A4 Enter `i. 6` and `#p` and `i. #p` and `i. -#p`. Then state the meanings of the primitives `#` and `i.` .

A5 Enter `asp=. p*_1^i. #p` to get a list of primes that alternate in sign (enter `asp` alone to display them). Compare the results of `-/p` and `+/asp` and state in English the significance of the phrase `-/` .

[`-/` yields the alternating sum of a list argument]

A6 Explore the assertion that `%/a` is the alternating product of the list `a`. [Use `arp=. p*_1^i. #p`]

A7 Enter the following expressions on the computer, and then state the purpose of the adverb `~` :

```
into=. %~  
4 into x=. i.5  
powerof=. ^~  
3 powerof x  
x +/ x  
+/~x  
plustable=. +/~@i.  
plustable 6
```

[The dyadic case of `f~` reverses the sense of `f`; in the monadic case, `f~ x` is equivalent to `x f x.`]

A8 Execute (by entering on the computer) each of the sentences of

the informal proof at the end of Section A to test the equivalences. Then annotate the sentences to state why each is equivalent to its predecessor.

A9 Experiment with, and comment upon, the following and similar sentences:

```
s=. '4%5'  
|.s  
do=. "  
do s  
do |.s  
|.i.5  
|. 'I saw'
```

[Enclosing quotes produce a list of characters, that may be manipulated like other lists and may, if they represent proper sentences, be *executed* by applying the verb ". .]

A10 Experiment with and comment upon:

```
]a=. <1 2 3  
>a  
2*a  
2*>a  
]b=. (<1 2 3) , (<'pqrs')  
|.b  
#b  
1 2 3; 'pqrs'
```

[< boxes its argument to produce a *scalar encoding*; > opens it.]

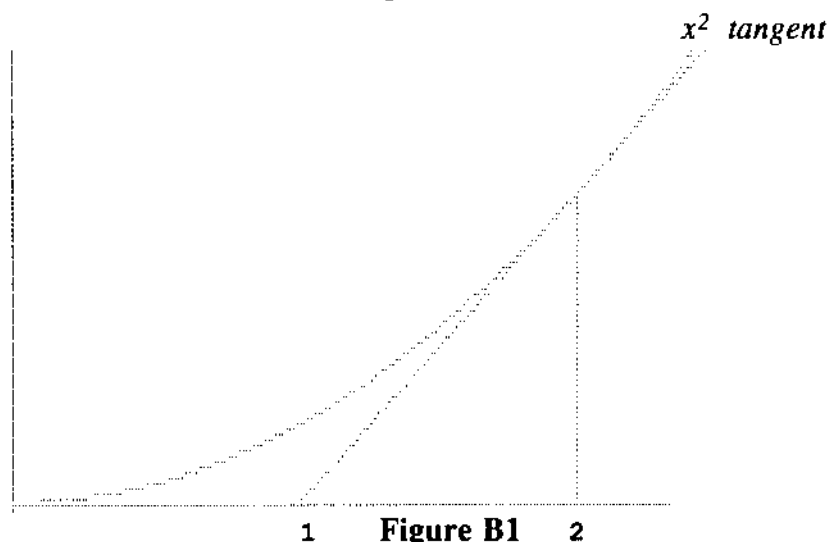
A11 Experiment with and comment upon:

```
f=. >:@i.  
f 4  
f  
g=. %@(+/% #)  
g f 4  
g  
linear=. 5!:5@<  
linear 'g'  
%@(+/% #)  
tree=. 5!:4@<  
tree 'g'
```

[Entering the name of a function alone shows its definition in *display* form; the *foreign* conjunction !: provides other forms]

B. DERIVATIVES, INTEGRALS, AND SECANT SLOPES

The central notions of the calculus are the *derivative* and the *integral* or *anti-derivative*. Each is an *adverb* in the sense that it applies to a function (or *verb*) to produce a *derived* function. Both are illustrated (for the square function x^2) by the graph in Figure B1, in which the slope of the tangent at the point x, x^2 as a function of x is the derivative of the square function, that is $2x$. The area under the graph is the integral of the square, that is, the function $x^3 / 3$, a function whose derivative is the square function.



Certain important properties of a function are easily seen in its graph. For example, the *square* has a *minimum* at the point $0, 0$, and increases to the right of zero at an accelerating rate.

These properties concern the *local* behaviour of a function in the sense that they concern how rapidly the function value is changing at any point. They are not easily discerned from the expression for the function itself, but *are* expressed directly by its derivative.

More surprisingly, a host of important functions can be defined simply in terms of their derivatives. For example, the important *exponential* (or *growth*) function is completely defined by the fact that it is equal to its derivative (therefore growing at a rate equal to itself), and has the value 1 for the argument 0.

The *difference* calculus is based upon *secant slopes*, such as illustrated in Figure B2. The slope of the *secant* (from *ligne secante*, or *cutting line*) through points $x, f\ x$ and $(x-r), (f\ x-r)$ is obtained by dividing the *rise* $(f\ x) - (f\ x-r)$ by the *run* r ; the result of $((f\ x) - f\ x-r) \% r$ (or $r \% \sim (f\ x) - f\ x-r$) is called the *r-slope* of f .

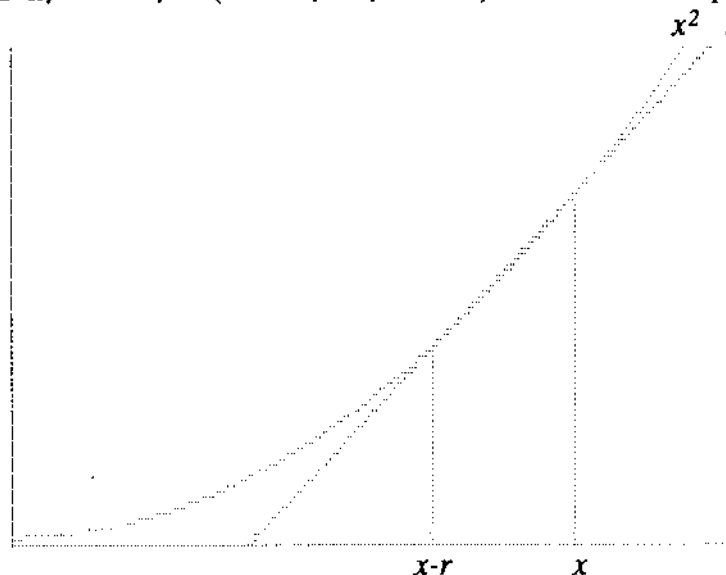


Figure B2

We will use the secant slope to illustrate methods of defining a conjunction; beginning with a sentence that defines the function that is to result from the conjunction, and using x . and y . to refer to the left and right arguments of the conjunction:

```
m=. '%&x. @ (] -&y. -&x.)'
```

We now apply the *explicit definition* conjunction represented by the colon to produce the desired slope conjunction, and illustrate its use as follows:

```
S=. m : 2
f=. ^&2
r=. 1
r S f x=. 1 2 3 4
1 3 5 7
0.1 S f x
1.9 3.9 5.9 7.9
```

A space must precede the colon

```
1e_8 S f x
2 4 6 8
```

It may be noted that the small increment $1e\_8$ in the last expression produces a good approximation to the slope of the tangent line, that is, to the derivative. An approximate derivative adverb can therefore be defined as follows:

```
AD=. 1e_8 $
  f AD x
2 4 6 8
  ^&3 AD x
3 12 27 48
```

An expression for the slope of the square function $f=. \wedge 2$ used above can be obtained directly by algebraically expanding the expression $f(x-r)$ to the equivalent form $(x^2)+(-2*x*r)+(r^2)$, as shown in the following *proof*, or list of identical expressions:

```
((f x)-(f x-r))%r
r%~ (f x)-f x-r
r%~ (f x)-((x^2)+(-2*x*r)+(r^2))
r%~ (x^2)-((x^2)+(-2*x*r)+(r^2))
r%~ 0-(-2*x*r)+(r^2)
r%~ (2*x*r)-(r^2)
(2*x)-r
```

Moreover, if r is set to zero in the final expression $(2*x)-r$, the result is $2*x$, the value of the derivative of $\wedge 2$.

Similar analysis can be performed on other power functions. For example, if $g=. \wedge 3$:

```
r%~ (g x)-g x-r
r%~ (3*(x^2)*r)+(-3*x*r^2)+r^3
(3*x^2)+(-3*x*r)+r^2
```

Again the derivative is obtained by setting r to zero, leaving $3*x^2$.

If $pq=. p+q$, the function pq is the sum of the functions p and q , and any r -slope of pq is therefore the sum of the r -slopes of p and q . Thus:

```
r $ pq x
r%~ (pq x)-pq x-r
r%~ ((p x)+(q x)) - ((p x-r)+(q x-r))
r%~ ((p x)-(p x-r)) + (q x)-q x-r
(r%~ (p x)-p x-r) + (r%~ (q x)-q x-r)
(r $ p x) + (r $ q x)
((r $ p) + (r $ q)) x
```

Again the limiting case of a zero value of x gives the derivative of pq (that is, $p+q$) as the sum of the derivatives of p and q :

$$\begin{aligned} p + q \text{ AD } x \\ (p \text{ AD } + q \text{ AD}) x \end{aligned}$$

A similar identity can be developed for the product or *times* function $t=.$ $p*q$:

$$\begin{aligned} (x \text{ S } t) x \\ ((p*x \text{ S } q)+(x \text{ S } p*q)-(x^0*x \text{ S } p*x \text{ S } q)) x \end{aligned}$$

The limiting case of the derivative is obtained by dropping the final term (that includes multiplication by x):

$$\begin{aligned} p * q \text{ AD } x \\ ((p * q \text{ AD}) + (p \text{ AD } * q)) x \end{aligned}$$

The difference calculus proves useful in a wide variety of applications, including approximations to arbitrary functions, and financial calculations in which events (such as payments) occur at fixed intervals.

We will also treat the *vector* calculus and the *fractional* calculus, usually met only in advanced courses.

B1 Enter the sentences of Section B, and confirm that the results agree with the text.

B2 Enter:

$$\begin{aligned} x &= .5 \\ r &= .0.1 \\ f &= .^4 \\ x \text{ S } f \text{ } x \end{aligned}$$

for the 0.1-slope of the fourth power at the point $x=.5$. Then write and enter an equivalent sentence for the slope that does not use the conjunction s .

$$\begin{aligned} [\text{ +/_1 4 _6 4 } * (x^0 \text{ 1 2 3}) * (x^3 \text{ 2 1 0}) \\ \text{ or} \\ \text{ +/_1 4 _6 4 } * (x^i \text{ .4}) * (x^i \text{ .-4})] \end{aligned}$$

B3 The expressions of Ex B2 will not work for a list argument x . Explore the following expressions:

$$\begin{aligned} vm &= .i.0] \text{ ^~/ } [\\ x &= .0 \text{ 1 2 3 4 5} \end{aligned}$$

```

x vm 4
r vm -4
(x vm 4)*(x vm -4)
_1 4 _6 4 * (x vm 4) * (x vm -4)
+/_1 4 _6 4 *(x vm 4) * (x vm -4)

```

B4 Enter the sentences:

```

x=. 0.1*i. 19
r=. 0.01
f=. ^&3
f x
r S f x

```

Then use the results to sketch graphs of both the function f and its slopes, and confirm that the slope values accurately reflect the slopes of the original function.

B5 Enter `sh1=. 4.1&"`: and use it to produce neater (and sufficiently precise) results for the graphs of Ex B4, as in `sh1 f x` and `sh1 r S f x`.

B6 Repeat Ex B4 for each of the following cases:

```

f=. (^&3)-(^&2)
f=. sin=. 1&o.
f=. cos=. 2&o.
f=. ^

```

B7 Comment on the results of the following sentences, and on analogous results for the general power ^{n} :

```

r=. 0.01
r S (r S (r S (^&3))) x=. 1 2 3 4 5 6
!3
A=. r S
^&3 A A A x
^&4 A A A A x
^&5 A A A A A x
^&5 A A A A A A x

```

B8 Write an expression for the r -slope of the sum of two functions (that is, $(r S (f+g))$) in terms of the slopes of the individual functions, and then state the result in English.

B9 Repeat the steps of Ex B8 for the product of two functions.

B10 Draw a rectangle and label the left and bottom sides with f and g to indicate their lengths. Inside it draw a horizontal line near the top and a vertical line near the right side, labelling the large inner rectangle with the lengths f and g .

Noting that the areas of the large rectangles represent the products $(f \cdot g)$ and $(f \cdot g) \cdot x$, relate the areas of all five rectangles to terms in the result of Ex B9.

B11 Enter $x = . 1 . 5$ and $] x$ and $1 : "0 x$ and $0 : "0 x$ to determine the behaviour of the functions $] x$ and $1 :$ and $0 :$.

[$] x$ is the identity function; $1 :$ and $0 :$ are constant functions that yield 1 and 0 respectively.]

B12 Enter $r s] x$ to determine the r -slope of the identity function, and $r s (x s]) x$ to determine *its* slope. Then define functions d and e that are equivalent to the functions $r s]$ and $r s (x s])$.

[$d = . 1 : "0$ and $e = . 0 : "0$]

C. THE CONJUNCTION D.

We begin not with an adverb for the derivative, but with a *conjunction* D , whose right argument specifies the *order* of the derivative. This section provides brief examples of the topics elaborated in subsequent chapters:

<code>cube = . ^&3 " 0</code>	Cube of rank 0 (applies to scalars)
<code>x = . 0 1 2 3 4</code>	
<code>cube x</code>	
<code>0 1 8 27 64</code>	
<code>cube D. 1 x</code>	(First) derivative of cube
<code>0 3 12 27 48</code>	
<code>g = . (3 : * ^&2) "0</code>	The derivative of the cube is 3 times the square
<code>g x</code>	
<code>0 3 12 27 48</code>	
<code>sdc = . cube D. 2</code>	The second derivative of the cube
<code>sdc x</code>	
<code>0 6 12 18 24</code>	
<code>(6 : *) "0 x</code>	Six times the identity function
<code>0 6 12 18 24</code>	
<code>cube D.1 D.1 x</code>	
<code>0 6 12 18 24</code>	

```
cube D.3 x
6 6 6 6 6
```

The constant function 6

```
6:"0 x
6 6 6 6 6
```

```
integral=. cube I
```

The integral adverb defined in §2C

```
integral x
0 0.25 4 20.25 64
```

```
(^&4 % 4:) x
0 0.25 4 20.25 64
```

```
integral D. 1 x
0 1 8 27 64
```

The derivative of the anti-derivative
of the cube is the cube

The derivative illustrated above concerned a function applied to *scalars*; that is, the rank zero function x^3 . The *vector calculus* concerns functions that apply to *vector* (or *rank 1*) arguments, and the derivative adverb `D. 1` then produces a function that yields the derivative (rate of change) of the function with respect to *each* element of the vector argument. For example:

```
d=. 2 3 4
vol=. */"1
vol d
24
```

A volume function that applies to vectors
Volume of box having dimensions `d`

```
vol D. 1 d
12 8 6
```

Rates of change when each length is
changed while others remain constant

The function `*/` can also be applied to a *table* of dimensions (having rank 2). Thus:

```
td=. 2 3 $ 1 2 3 2 3 4
td
1 2 3
2 3 4
```

Table of dimensions

```
vol td
6 24
```

List of volumes (because `vol` has
rank 1)

```
vol D. 1 td
6 3 2
12 8 6
```

Table of derivatives

```
*/ td
2 6 12
```

Not a vector function; products are
over columns, not rows

```

    */ D.1 td
2 0 0
0 3 0
0 0 4

```

Not a vector derivative

```

1 0 0
0 2 0
0 0 3

```

We will now define a *vector derivative* adverb that first assigns rank 1 to the function, and then applies the derivative:

```

    VD=. ("1) (D. 1)
    */ VD td
6 3 2
12 8 6

```

Similarly, a function intended to apply to scalars must be assigned rank 0. Thus:

```

    ^&3"0 D. 1 x
0 3 12 27 48
    D=. ("0) (D.1)

```

Scalar derivative

```

    ^&3 D x
0 3 12 27 48

```

In treatments of the calculus that do not use the notion of vector functions, the dimension *d* would be treated as the catenation of the coordinates *x*, *y*, and *z*, and the derivatives would be referred to as *partial derivatives with respect to x, y, and z*, respectively.

The function `VOLS=. */\ "1` is an example of a vector function that yields a *vector* result; its derivative yields the rate of change of *each* element of its result with respect to *each* element of its argument. For example:

```

    vol=. */ "1
    VOLS=. */\ "1
    vol d=. 2 3 4
24
    vol VD d
12 8 6

```

```

    VOLS d
2 6 24
    VOLS VD d
1 3 12
0 2 8
0 0 6

```

The idea of a fractional derivative may appear absurd, but so does the idea of a fractional power (such as $x^{0.5}$ for the square root of x) when one begins with a definition of the form $x^{1/n} = m$ for the expression m^n . Similarly, the extension of the factorial function to fractional arguments (as in the *gamma* function) may seem absurd when based upon the simple definition for integers, that is, $n! = n \cdot (n-1) \cdot \dots \cdot 1$.

The following quote (from page 161 of *The Fractional Calculus* [6]) provides some indication of its potential utility:

"... the fractional calculus enables a plethora of transcendental functions to be expressed in terms of a basis set of only three functions: the inverse binomial, the exponential, and the zero-order Bessel."

For example, using the fractional derivative conjunction $\mathcal{D}$ defined in Section B of Chapter 5:

$f = 1$	The constant function <i>one</i>
$y = 1 \ 2 \ 3$	
$\mathcal{D}^{0.5} y$	Semiderivative of a constant function
0.56419 0.398942 0.325735	
$\% \pi: 0. y$	Expression for equivalent from [6]
0.56419 0.398942 0.325735	
$y^{(0)} \mathcal{D}^{0.5} y$	Semi-derivative of the identity function
1.12838 1.59577 1.95441	
$+: \pi: y^{(0)}$	Equivalent double square-root of y over π from [6]
1.12838 1.59577 1.95441	
$\mathcal{D}^{0.5} y^3$	Semiderivative of cube
1.79416 10.1493 27.9682	
$(\% \pi) * (y^{(0)})^3 \mathcal{D}^{0.5} y$	
1.80541 10.2129 28.1435	
$\mathcal{D}^{0.5} y^3$	Semi-integral of cube
0.515835.83596 24.123	
$(\% \pi) * ((y^{(0)} + 0.5)^3) \mathcal{D}^{0.5} y$	
0.515835.83596 24.123	

C1 Enter the sentences of Section C, and confirm that the results agree with the text.

C2 Review Ex B2, and use the results to define a function equivalent to the derivative $x^4 D$.

[For small values of the increment x , terms in the expression for the slope that involve a non-zero power of x can be ignored, leaving the function $4x^3$ for the derivative.]

C3 Use Ex B4 to define a function for the derivative of x^n .

C4 Define a function equivalent to the second derivative of x^5 .

[$x^5 D D$ is equivalent to $5 \cdot 4 \cdot x^3$]

C5 Review Exercises B8,9, and define expressions for the derivative of the sum and product, $f+g$ and $f \cdot g$.

C6 Apply your result in Ex C5 to the product of the functions $f = x^2$ and $g = x^3$, and test the assertion that its derivative agrees with the derivative of x^5 .

C7 Comment on the results of the following sentences:

```
([] * []) x = . 0 1 2 3 4 5
^2 x
^2 D x
([] * []) D x
(([] * []) D) + ([] D * []) x
([] + []) x
2x x
+: x
```

[This is an informal proof (based upon the expression for the derivative of a product) that the derivative of the square function is the double function]

C8 Use the fact that x^3 is equivalent to $x \cdot x^2$ together with the results of Ex C7 to obtain the derivative of the cube function.

C9 Comment on the following sentences:

```
f = . ^3
c = . 4
c * f x = . 0 1 2 3 4 5
(c"0 * f) x
(c"0 * f) D x
(c"0 * f D) x
```

[The derivative of a constant multiple of a function f
is the constant times the derivative of f]

C10 Using $\text{VD} = .$ ("1") (D.1) and each of the functions f defined below, predict and confirm the results of the expressions $f \ x = .$ 2 3 4 and $f \ \text{VD} \ x$:

$f = . \ +/$	$f = . \ +/\backslash$	$f = . \ +/\backslash .$
$f = . \ .$	$f = . \ . \backslash$	$f = . \ . \backslash .$
$f = . \ 1\&A .$	$f = . \ 1\&A . \backslash$	$f = . \ 1\&A . \backslash .$

C11 Define the determinant function $\text{det} = . \ -/ . \ *$ and the matrix derivative $\text{MD} = .$ ("2") (D.1) . Then predict and confirm the results of $\text{det} \ m = . \ *$: i. 3 3 and $\text{det} \ \text{MD} \ m$.

C12 Define a function area such that $\text{area} \ d = .$ 2 3 4 yields the surface area of a box of dimensions d . Predict the result of the derivative of the area function, and enter $\text{area} \ \text{VD} \ d$ to confirm your prediction.

[$\text{area} = . \ +/@+:@(1\&(*/\backslash.))$]

D. INTERPRETATIONS

Although the function vol was completely defined by the expression $\text{vol} = . \ */"1$, we added (in Section C) the *interpretation* that $\text{vol} \ d$ yielded the *volume* of a box of dimensions d . However, to anyone familiar with finance and fearful of geometry, the equivalent definition $\text{cost} = . \ */"1$, and the interpretation of the result of $\text{cost} \ d$ as the cost of two crates of three items each costing four dollars would be much more helpful.

In short, the definition of a function alone suffices; a given interpretation (among many possible) may provide either clarification or confusion, depending upon the background experience of the reader. Consequently, we will proceed by defining further functions without interpretation, deferring explicit interpretations until the reader has had an opportunity to provide her own.

Nevertheless, imaginative interpretations are important in gaining insight. We will illustrate this with two examples:

1. Enter the function path shown below, and experiment with the expression $a \ \text{path} \ b$ for various values of the scalars a and b . Then experiment with the derivative of the function $a\&\text{path} \ b$

expressions of the form `a$path D b`. Finally, before reading further, try to provide one or more interpretations of the function and of its derivative:

```
path=. (cos,sin)@*"0 [. (cos=.2&o.) [. (sin=.1&o.)
```

2. Define `PATH=. ((.0[ path ]) , (.0[ * ]))"1 0`. Experiment with it, and provide interpretations.

INTERPRETATIONS

```
0.1 path 2
0.980067 0.198669
```

Position of a car after 2 seconds, moving on a circular path at an angular velocity of 0.1 radians per second.

```
f=. 0.1&path
f 2
0.980067 0.198669
```

The monadic function with an angular velocity of 0.1 radians per second.

```
f D 2
_0.0198669 0.0980067
```

Velocity of motion at 2 seconds; that is, the speed in each of the two directions. The derivative of a function such as `f` is called its *gradient*.

```
f D"0 i. 3
_5.00002e_9 0.1
_0.00998335 0.0995004
_0.0198669 0.0980067
```

```
0.1 5 PATH 2
0.980067 0.198669 10
```

Flight path of an aeroplane climbing in a circle at 5 metres per second.

```
F=. 0.1 5&PATH
F 2
0.980067 0.198669 10
F D 2
_0.0198669 0.0980067 5
```

The velocity (speed in each of three directions).

```
F D D 2
_0.00980068 _0.0019867 0
```

The acceleration. Because the vertical speed is constant, the vertical acceleration is zero.

A separate chapter will be devoted to the matter of interpretations: they are important to an understanding of the calculus; they must draw upon a wide range of topics; and it is important to appreciate their limitations.

D1 Enter the sentences of §D; experiment with similar sentences.

E. UTILITIES

A programming language is said to be *general-purpose* if it can be used effectively in a wide range of applications. Such generality is not normally achieved by cluttering the language with facilities that directly cover all areas of application, but rather by providing a lean set of primary facilities, or *primitives*, that include the power to define and name secondary and tertiary facilities suited to different disciplines and sub-disciplines. These user-defined facilities will be called *utilities*.

We now present collections of utilities that will be used throughout the text. Their definitions are designed for exposition, and can be made more efficient; for example, by using *amend* (1) instead of *membership* (•) in the definition of *vs*. The reader may choose to examine them at this point, or may skip them and return to them when they occur in later topics. He may also choose to assign fixed *given* names to some of them as discussed below.

Given Names

A given name is like a child's given name; once assigned, it cannot be re-assigned. However, it can be expunged, and then be re-used. In entering a utility it should be assigned a given name. Thus:

```

      rFd=. o.@(%&180)      Radians from degrees
      SIN:=. 1&o.@rFd      Given name for sine in degrees
      SIN: 0 45 90
0 0 0.707107 1
      SIN:=. ]
not reassignable
      rFd=. ]
      SIN: 0 45 90
0 0.850904 0.893997
```

A change in the meaning of a given name due to a change in a component function can be avoided by using given names for component functions. It can also be avoided by using the adverb *£*. to *fix* a definition, replacing each component name by its referent. Thus:

```

      SIN:                               Display of function definition
```

1	&	o.	@	rFd
---	---	----	---	-----

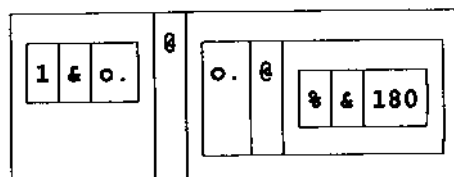
```
rFd=. o.@(%&180)
4!:56 <'SIN:'
```

Restore definition of `rFd`
 Expunge given name

1

```
SIN=. 1&o.@rFd f.
SIN:
```

Fix definition of `SIN`:



Housekeeping Utilities

Entering sentences on a computer can be simplified by using any available editing facilities. These differ from computer to computer, but usually include the ability to erase by backspacing, to insert and delete by using the keys (marked with left and right arrows) to move the cursor, and to recall earlier entries for editing and re-entry. On some systems the key marked F1 will evoke detailed instructions for editing.

The current names established can be saved (together with their referents) and can be re-copied at any later time by using the following utilities:

```
save=. 2!:2@< [ . load=. 2!:4@<
```

The arguments of `save` and `load` are file names, as in `save 'abc'` and `load 'abc'`. The function:

```
clix=. ('unused';';'noun';'verb';'adverb';'conj')&i.
```

provides the *class index* of its argument. For example, `clix <'noun'` is 2, and `clix 'conj';'verb'` is 5 3. These class indices are used as arguments to the function:

```
names =. 4!:1
```

which yields the list of (boxed) names in the class or classes specified by its argument. Thus, `names 2 3` yields the list of (pro)noun and (pro)verb names, as does `names clix 'noun';'verb'`.

The function `class=. 4!:0@<` yields the class indices of a list of boxed names to which it is applied. Hence, `clix^:_1 class` yields the corresponding names of the classes. For example, `clix^:_1 class 'clix'` yields the boxed name `'verb'`.

The ranks of a function f are given by the expression $f \text{ b. } 0$. For example, $+ \text{ b. } 0$ is $\_ 0 \ 0$, and $\# \text{ b. } 0$ is $\_ 1 \ \_$, and $(+ / * \#) \text{ b. } 0$ is $\_ \_ \_$.

The display of a function is obtained by entering the function name alone without an argument. Alternative displays are provided by the adverb `display=.` '5!;x. @<' : 1. For example:

```
4 display 'save'          5 display 'save'
_ @ _ 2                    2!;2@<
_ @ _ 2
```

Circular and Hyperbolic functions.

`sin=.` 1&o. `sinh=.` 5&o.

`cos=.` 2&o. `cosh=.` 6&o.

`tan=.` 3&o. `tanh=.` 7&o.

`SIN=.` `sin@rfd` Sine in degrees

`COS=.` `cos@rfd`

`TAN=.` `tan@rfd`

`rfd=.` o.@(%180) Radians from degrees

Matrix product and linear functions. The dot conjunction applied to the sum and product functions yields a function commonly referred to as the *dot* or *matrix* product. Thus:

```
mp=. +/ . *
]m=. i. 3 3
0 1 2
3 4 5
6 7 8

]n=. i. 4 3
0 1 2
3 4 5
6 7 8
9 10 11

n mp m
15 18 21
42 54 66
69 90 111
96 126 156

3 2 1 mp m
12 18 24
1 4 6 mp m
48 59 70
```

Left and right bonds of the matrix product distribute over addition; that is, $a \text{ mp } c + d$ is $(a \text{ mp } c) + (a \text{ mp } d)$, and $mp \&b \ c + d$ is $(mp \&b \ c) + (mp \&b \ d)$. For example:

```
mp&m 3 2 1 + 1 4 6      (mp&m 3 2 1) + (mp&m 1 4 6)
60 77 94                60 77 94
```

A function that distributes over addition is said to be *linear*; the name reflects the fact that a linear function applied to the coordinates of collinear points produces collinear points. For example:

```

]line=. 3 _7 1,:2 2 4      ]a=. 3 1,:_4 2
3 _7 1                    3 1
2 _2 4                    _4 2

a&mp line
11 _19 7
_8 _32 4

mp& 3 1 _2 line      mp&3 1 _2 a &mp line
0 0                  0 0

```

Polynomials. An atomic constant multiplied by an integer power (as in a^n) is called a *monomial*, and a sum of monomials is called a *polynomial*. We now define a polynomial function, the items of its list left argument being called the *coefficients* of the polynomial:

```
p=. +/@([ * ] ^ i.@#@[] " 1 0
```

For example:

```

c=. 1 2 3 [ x=. 0 1 2 3 4
c p x      1 3 3 1 p x
1 6 17 34 57      1 8 27 64 125

```

The polynomial may therefore be viewed as a weighted sum of powers, the weights being specified by the coefficients. It is important for many reasons. In particular, it is easily expressed in terms of sums, products, and integral powers; it can be used to approximate almost any function of practical interest; and it is *closed* under a number of operations; that is, the sums, products, derivatives, and integrals of polynomials are themselves polynomials. For example:

```

x=. 0 1 2 3 4 [ b=. 1 2 1 [ c=. 1 3 3 1
(b p x) + (c p x)      Sum of polynomials
2 12 36 80 150
b +/@,: c              "Sum" of coefficients
2 5 4 1
(b +/@,: c) p x      Sum polynomial
2 12 36 80 150
(b p x) * (c p x)      Product of polynomials
1 32 243 1024 3125

```

```

    b +//.0(*) c          "Product" of coefficients
1 5 10 10 5 1
    (b +//.0(*) c) p x    Product polynomial
1 32 243 1024 3125
    c&p D x                Derivative of polynomial
3 12 27 48 75
    derc=. }.0[] * i.0#)  "Derivative" coefficients
    derc c
3 6 3
    (derc c) p x          Derivative polynomial
3 12 27 48 75
    intc=. 0: , ] % >:0i.0# "Integral" coefficient
    intc c                derc intc c
0 1 1.5 1 0.25          1 3 3 1

```

The polynomial may also be defined more briefly in terms of the base-value function `#.` as follows:

```
p=. #.~&|. "1 0
```

A polynomial is "linear in its coefficients" in that $(c+d) \text{ p } x$ is $(c \text{ p } x) + (d \text{ p } x)$. This linearity can be made clear by expressing $c \text{ p } x$ as $m \text{ mp } c$, where m is the *Vandermonde* matrix obtained as a function of x and c . Thus:

```

vm=. [ ^/ i.0#0]
x=. 0 1 2 3 4 5
c=. 1 3 3 1
    x vm c          (x vm c) mp c          c p x
1 0 0 0          1 8 27 64 125 216          1 8 27 64 125 216
1 1 1 1
1 2 4 8
1 3 9 27
1 4 16 64
1 5 25 125

```

The expression `c=. (f x) % . x^/i.n` yields an n -element list of coefficients such that $c \text{ p } x$ is the best least-squares approximation to the values of the function f applied to the list x . In other words, the value of $+/\text{sqrr } (f x) - c \text{ p } x$ is the least achievable for an n -element list of coefficients c .

We now define a conjunction `FIT` such that `a FIT f x` produces the coefficients for the best polynomial fit of a elements:

```

FIT=. 'y. %. ^/&(i. x.)' : 2
]c=. 5 FIT ! x=. 0 1 2 3 4
1 _2.08333 3.625 _1.91667 0.375
  c p x
1 1 2 6 24      1 1 2 6 24
]c=. 4 FIT ! x
0.871429 3.27381 _3.71429 1.08333
  c p x
0.871429 1.51429 1.22857 6.51429 23.8714

```

Graphs. A barchart of a function can be produced by making a table of comparisons between a list of function values and an ordered list of the range of the same function values. For example:

```

]v=. c p x=. i. 7 [ c=. 0 6 _1
0 5 8 9 8 5 0
  grid=. >./@(- (>./-<./)@[*] %~ i.@>:@]
  v grid 10
9 8.1 7.2 6.3 5.4 4.5 3.6 2.7 1.8 0.9 0
  PL=. '.*'&PLOT=. {~
    (v grid 6) <:/ v
0 0 0 1 0 0 0
0 0 1 1 1 0 0
0 0 1 1 1 0 0
0 1 1 1 1 1 0
0 1 1 1 1 1 0
0 1 1 1 1 1 0
1 1 1 1 1 1 1
    PL (v grid 6) <:/ v
...*...
...***...
...***...
...***...
...*****...
...*****...
...*****...
*****

```

The function `</\` applied to a list of *zeros* and *ones* suppresses (i.e., replaces by a *zero*) each *one* following the first; applied to a barchart, it produces the corresponding graph by suppressing all but the first *one* in each column:

```

graph=. </\@ (grid <:/ [])
v graph 6
0 0 0 1 0 0 0
0 0 1 0 1 0 0
0 0 0 0 0 0 0
0 1 0 0 0 1 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
1 0 0 0 0 0 1
    ' + ' PLOT v graph 6
    +
    + +
    + +
    + +
    + +

```



```

      '.*' PLOT 6 10 *: VS ] 1 2 3 4
.....*
.....
.....*.....
.....
...*.....
*.....

```

The same plot with a dot for the background, and an asterisk for the plotted points.

The definition of the conjunction **vs** will now be developed in a series of illustrated steps:

```

f=. *: [. g=. ]
s=. 6 10 [ x=. 1 2 3 4

scale=. ]%>./"1 [. round=. <.@(0.5%+)
N=. 'scale@(((>./-])@x. ,: (]-<./)@y.)@))' : 12

s f N g x
1      0.8 0.466667 0
0 0.333333 0.666667 1

r=. [#. |: @ (round@ (<: @ [*scale@]))

s r s f N g x
50 43 26 9

(i. s) e. s r s f N g x
0 0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 0 0

VS=. 'i.@[ e. [ r x. N y.' : 12

s f VS g x
0 0 0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 0 0

```

The conjunction **vs** is used in §6B in the plot of sine versus cosine to produce a circle, and in the plot of the hyperbolic sine versus the hyperbolic cosine to produce an hyperbola.

Inverse, Reciprocal, and Parity. We will now define and illustrate the use of four further adverbs:

I=. ^: _1	Inverse adverb
R=. %@	Reciprocal adverb
ODD=. .: -	Odd adverb
EVEN=. .: -	Even adverb
*: I x=. 0 1 2 3 4 5	Inverse of the square,
0 1 1.41421 1.73205 2 2.23607	that is, the square root
*: R x	Reciprocal of the square,
_ 1 0.25 0.111111 0.0625 0.04	that is, %@*: , or ^&_2
c=. 4 3 2 1	
even=. c&p EVEN	Even part of polynomial c&p
odd=. c&p ODD	Odd part of polynomial
even x	Even function applied to x
4 6 12 22 36 54	
odd x	Odd function applied to x
0 4 14 36 76 140	
(even + odd) x	Sum of even and odd parts
4 10 26 58 112 194	is equal to the original
c&p x	function c&p
4 10 26 58 112 194	
4 0 2 0 p x	Even part is a polynomial with non-
4 6 12 22 36 54	zero coefficients for even powers
0 3 0 1 p x	Odd part is a polynomial with non-
0 4 14 36 76 140	zero coefficients for odd powers

For an even function, $f -y$ equals $f y$; for an odd function, $f -y$ equals $-f y$. The adverb GR can be used to show the graphic properties of the even and odd parts of a function: the graph of the even part is "reflected" in the vertical axis, and the odd part in the origin.

Binomial Coefficients. $m!n$ is the number of ways that m things can be chosen out of n ; for example $2!3$ is 3, and $3!5$ is 10. The expression $c=. (i. n+1)!n$ yields the *binomial coefficients of order n*, and $c p x$ is equivalent to $(x+1)^n$. For example:

```
]c=. (i. n+1)!n=. 3
1 3 3 1
```

```

c p x=. 0 1 2 3 4 5
1 8 27 64 125 216
(x+1) ^ n
1 8 27 64 125 216
<@ (i.@>: ! j)"0 i. 6

```

1	1	1	1	2	1	1	3	3	1	1	4	6	4	1	1	5	10	10	5	1
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	----	----	---	---

Complex Numbers. Just as subtraction and division applied to the *counting* numbers (positive integers) introduce new classes of numbers (called *negative* numbers and *rational* numbers), so does the square root applied to negative numbers introduce a new class called *imaginary* numbers. For example:

```

a=. 1 2 3 4 5 6
]b=. -a
_1 _2 _3 _4 _5 _6
% a
1 0.5 0.333333 0.25 0.2 0.166667
% b
0j1 0j1.41421 0j1.73205 0j2 0j2.23607 0j2.44949

```

Negative numbers

Rational numbers

Imaginary numbers

Arithmetic functions are extended systematically to this new class of numbers to produce *complex* numbers, which are represented by two real numbers, a *real* part and an *imaginary* part, separated by the letter *j*. Thus:

```

a+%:b
1j1 2j1.41421 3j1.73205 4j2 5j2.23607 6j2.44949
j. a
0j1 0j2 0j3 0j4 0j5 0j6
]d=. a+j. 5 4 3 2 1 0
1j5 2j4 3j3 4j2 5j1 6
+d
1j_5 2j_4 3j_3 4j_2 5j_1 6
d*d
26 20 18 20 26 36
%: d*d
5.09902 4.47214 4.24264 4.47214 5.09902 6
|d
5.09902 4.47214 4.24264 4.47214 5.09902 6

```

Complex numbers

The function *j.* multiplies its argument by *0j1*

The monad *+* is the *conjugate* function; it reverses the sign of the imaginary part of its argument

Product with the conjugate produces a real number

The *magnitude* of a complex number

E1 Enter the expressions of Section E, and verify that the results agree with those given in the text.

E2 Predict the results of each of the following sentences, and then enter them to validate your predictions:

```
D=. ("1) (D.1)
x=. 1 2 3 4 5
|. D x
2%|. D x
3 1 0 2 &{ D x
+/\ D x
+/\. D x
```

E3 Define `show=. {&'.'*` and use it to display the results of Exercises E2, as in `show |. D x`.

E4 Define a function `rFd` to produce radians from degrees, and compare `rFd 90 180` with `o. 0.5 1`.

```
[ rFd=. %&180@o. ]
```

E5 Define a function `AREA` such that `AREA v` yields the area of a triangle with two sides of lengths `0{v` and `1{v` and with an angle of `2{v` degrees between them. Test it on triangles such as `2 3 90` and `2 3 30`, whose areas are easily computed.

```
[ AREA=. -:@ (0&{ * 1&{ * 1&o.@rFd@(:)"1 ]
```

E6 Experiment with the vector derivative of the triangle area function of Exercise E5, using `VD=. ("1) (D.1)`.

```
[ AREA VD 2 3 90 ]
```

E7 Heron's formula for the area of a triangle is the square root of the product of the semiperimeter with itself less each of the three sides. Define a function `hat` to give Heron's area of a triangle, and experiment with its vector derivative `hat VD`. In particular, try the case `hat VD 3 4 5`, and explain the (near) zero result in the final element.

```
[ hat=. %:@(*)@(-:@(+/-) - 0: , )]"1 ]
```

E8 Define a function `bc` such that `bc n` yields the binomial coefficients of order `n`, a function `tbc` such that `tbc n` yields a table of all binomial coefficients up to order `n`, and a function `tabc` for the corresponding alternating binomial coefficients.

```

[ bc=. i.0>: ! ]
tbc=. !/~ 0 (i.0>:)
tabc=. *.0tbc    ]

```

E9 Test the assertion that $(bc \ n) \ p \ x = i. \ 4$ is equivalent to x^{n+1} for various values of n .

E10 Write an expression to yield the matrix m such that $mpsm$ is equivalent to a given linear function L , and test it on the linear functions $L = . \ | \ . \ "1$ and $L = .36A. \ "1$, using the argument $x = . \ 2 \ 3 \ 5 \ 7 \ 11$.

```

[ L = x ]

```

E11 Experiment with the use of various functions on imaginary and complex numbers, including the exponential, the sine, cosine, hyperbolic sine and hyperbolic cosine. Also experiment with matrices of complex numbers and with the use of the matrix inverse and matrix product functions upon them.

F. CONCLUSION

We conclude this chapter with a brief statement of the ways in which the present treatment of the calculus differs from most introductory treatments:

1. *Partial* derivatives are treated in a simpler and more general way that stems from the use of functions that deal with arguments and results of arbitrary rank; this in contrast to the restriction to *scalars* (single elements) common in elementary treatments of the calculus.
2. *Fractional* derivatives constitute a powerful tool that is seldom treated in calculus courses. They are an extension of derivatives of integral order, introduced here in a manner analogous to the extension of the power function to fractional exponents, and the extension of the *factorial* and *binomial coefficient* functions to fractional arguments.
3. The *Calculus of Differences* is developed as a topic of interest in its own right rather than as a brief way-station to integrals and derivatives.
4. Few formal proofs are presented, and proofs are instead treated (as they are in *Arithmetic* [1]) in the spirit of Lakatos in his *Proofs and Refutations* [2], of which the author says:

“Its modest aim is to elaborate the point that informal, quasi-empirical, mathematics does not grow through the monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.”

5. As illustrated at the end of §1A, informal proofs will be presented by writing a sequence of expressions to imply that each is equivalent to its predecessor, and that the last is therefore equivalent to the first.

6. The notation used (as described in Section A) is unambiguous and executable. Because it is executable, it is used for experimentation; new notions are first introduced by leading the student to see them in action, and to gain familiarity with their use before analysis is attempted.

7. The exercises are an integral part of the development, and should be attempted as early as possible, perhaps even before reading the relevant sections. Try to provide (or at least sketch out) answers without using the computer, and then use it to confirm your results.

8. Two significant parts may be distinguished in treatments of the calculus:

a) A *body* comprising the central notions of *derivative* and *anti-derivative* (integral), together with their important consequences.

b) A *basis* comprising the analysis of the notion of *limit* (that arises in the transition from the secant slope to the tangent slope) needed as a foundation for an axiomatic deductive treatment.

The common (and perhaps logical) approach is to treat the basis first, and the body second. For example, in Johnson and Kiokemeister *Calculus with analytic geometry* [7], the section entitled *The derivative of a function* occurs after eighty pages of preliminaries.

The present text defers discussion of the analytical basis to Chapter 8, first providing the reader with experience with the derivative and the importance of its fruits, so that she may better appreciate the point of the analysis.

Chapter 2: DIFFERENTIAL CALCULUS

A. INTRODUCTION

This chapter will treat the derivatives of scalar (rank 0) functions, using the ideas introduced in Sections B and C of Chapter 1. In particular, we will use the general derivative D , and the scalar derivative adverb $D=$. ("0") ($D.1$) defined in terms of it. For example:

```
f4=. ]*f3=. ]*f2=. ]*f1=. ]*f0=. 1:"0
x=. 0 1 2 3 4
x^4
0 1 16 81 256
f4 x          f4 D x          (4:*f3) x
0 1 16 81 256 0 4 32 108 256 0 4 32 108 256
f3 x          f3 D x          (3:*f2) x
0 1 8 27 64   0 3 12 27 48   0 3 12 27 48
f2 x          f2 D x          (2:*f1) x
0 1 4 9 16    0 2 4 6 8      0 2 4 6 8
f1 x          f1 D x          (1:*f0) x
0 1 2 3 4     1 1 1 1 1      1 1 1 1 1
f0 x          f0 D x          0:"0 x
1 1 1 1 1     0 0 0 0 0      0 0 0 0 0
```

The derivatives of the power functions can be summarized in the single relation: $x^n D$ is $n \times x^{(n-1)}$. We will illustrate the important results for the derivatives of the sum $f+g$ and product $f \times g$ derived in Section B of Chapter 1, using $f=.$ $f3$ and $g=.$ $f4$ as follows:

```
(f+g) x
0 2 24 108 320
(f+g) D x
1e_12 7.00001 44.0001 135 304
(f D + g D) x
1e_12 7.00001 44.0001 135 304
(f*g) D x
1e_36 7.00002 448.001 5103.02 28672.1
((f * g D)+(f D * g)) x
0 7.00001 448.001 5103.01 28672
^&(3+4) D x
0 7 448 5103 28672
```

A1 Experiment with the expressions of this section.

B. TABLE OF DERIVATIVES

Table B1 lists a number of important functions, together with their derivatives. Each relation is accompanied by a phrase and an index that will be used to refer to it, as in *Theorem 2* or $\theta 2$ (where θ is the Greek letter *theta*) .

θ	NAME	FUNCTION	DERIVATIVE
1	Constant	a^0	$0 : ^0$
2	Identity	$]^0$	$1 : ^0$
3	Constant Times	$a \& *$	a^0
4	Sum	$f+g$	$f \text{ D} + g \text{ D}$
5	Difference	$f-g$	$f \text{ D} - g \text{ D}$
6	Product	$f * g$	$(f * g \text{ D}) + (f \text{ D} * g)$
7	Quotient	$f \% g$	$(f \% g) * (f \text{ D} \% f) - (g \text{ D} \% g)$
8	Composition	$f @ g$	$f \text{ D} @ g * g \text{ D}$
9	Inverse	$f \text{ I}$	$1 : \% f \text{ D} @ \text{ I}$
10	Reciprocal	$\% @ f$	$0 : - f \text{ D} \% f * f$
11	Power	$\wedge \& n$	$n^0 * \wedge \& (n-1)$
12	Monomial	$a^0 * \wedge \& n$	$a^0 * n^0 * \wedge \& (n-1)$
13	Polynomial	$c \& p$	$(d \text{ c}) \& p$

LEGEND: Functions f and g

Atomic constants a and n

List constant c

Polynomial function $p = . \# . \sim \& | . ^1 0$

Polynomial derivative $d = . \} . @ () * i . @ \#$

Adverb $D = . (^0) (D . 1)$ is the first derivative

Inverse adverb $I = . ^ : _1$

Table B1

Although analysis will be deferred to Chapter 8, we will here present arguments for the plausibility of the theorems:

- $\theta 1$ Since $a \neq 0$ x is a for any x , the rise is the *zero* function 0 :
- $\theta 2$ Since $(\uparrow h + x) - \downarrow x$ is $(h + x) - x$, the rise is h , the slope is $h \ast h$
- $\theta 3$ Multiplying a function by a multiplies all of its rises, and hence its slopes, by a as well.
- $\theta 4,5$ The rise of the function $f + g$ (or $f - g$) is the sum (or difference) of the rises of f and g . Also see the discussion in §1B.
- $\theta 6$ If the result of f is fixed while the result of g changes, the result of $f \ast g$ changes by f times the change in g ; conversely if f changes while g is fixed. The total change in $f \ast g$ is the sum of these changes. Also see the discussion in §1B.
- $\theta 7$ If $h = \downarrow f \ast g$, then $g \ast h$ is f , and, using $\theta 6$:
- $$\begin{aligned} f \ D \\ (g \ast h) \ D \\ (g \ast h \ D) + (g \ D \ast h) \end{aligned}$$
- The equation $f \ D = (g \ast h) \ D + (g \ D \ast h)$ can be solved for $h \ D$, giving the result of $\theta 7$.
- $\theta 8$ The derivative of $f \circ g$ is the derivative of f “applied at the point g ” (that is, $f \ D \circ g$), multiplied by the rate of change of the function that is applied first (that is, $g \ D$)
- $\theta 9$ $f \circ (f \ I) \ D$ is the product $f \ D \circ (f \ I) \ast f \ I \ D$ (from $\theta 6$). But since $f \circ (f \ I)$ is the identity function, its derivative is 1 : and the second factor $f \ I \ D$ is therefore the reciprocal of the first.
- $\theta 10$ Can be obtained from $\theta 7$ using the case $f = \downarrow$.
- $\theta 11$ Since $\wedge^5$ is equivalent to the product function $\downarrow \ast \wedge^4$, its derivative may be obtained from $\theta 6$ and the result for the derivative of $\wedge^4$. Further cases may be obtained similarly; that is, by *induction*.
- $\theta 12$ This follows from $\theta 3$ and $\theta 11$.
- $\theta 13$ As seen in the definition of the polynomial function p , it is a sum of monomials, the k th of which has the multiplier $a = \downarrow k \{c$. The final result is therefore as shown.

B1 Enter $f = .^{\wedge}2$ [, $f = .^{\wedge}3$ [, $x = .1$ 2 3 4, and then test the equivalence of the functions in the discussion of θ 7 by entering each followed by x , being sure to parenthesize the final fork.

B2 If c is a noun (such as 2.7), then $c^{\wedge}0$ is a constant function. Prove that $((c^{\wedge}0 + f) D = f D)$ is a tautology.

C. INTEGRAL OR ANTI-DERIVATIVE

As indicated in §1B, the *integral* or *anti-derivative* $\int$ is an adverb such that $f \int x$ yields the area under the graph of f from 0 to x . The fact that the derivative of the function $f \int$ is equal to f can be seen in Figure C1; since the difference $(f \int x+h) - (f \int x)$ is approximately the area of the rectangle with base h and altitude $f x$, the secant slope of the function $f \int$ is approximately f . Moreover, the approximation approaches equality for small h .

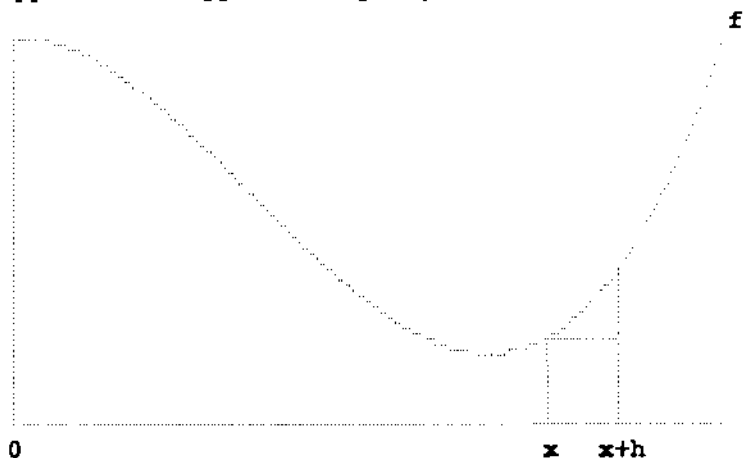


Figure C1

Figure C1 can also be used to suggest a way of approximating the function $\text{AREA} = f \int$; if the area under the curve is broken into n rectangles each of width x/n , then the area is approximately the sum of the areas of the rectangles with the common base h and the altitudes $f h^*i . n$. For example:

```
h=. y % n=. 10 [ y=. 2
cube=. ^3
cube h*i . n
0 0.008 0.064 0.216 0.512 1 1.728 2.744 4.096 5.832
```

```

+ / h * cube h * i . n
3.24
(4: %~ ^&4) y
4

```

The approximation can be improved by taking a larger number of points, but it can also be improved by using the areas of the *trapezoids* of altitudes $f h^*k$ and $f h^*k+1$ (and including the point h^*n). Since the area of each trapezoid is its base times the average of its altitudes, and since each altitude other than the first and last enter into two trapezoids, this is equivalent to multiplying the altitudes by the *weights* $w = .05, (1 \# \sim n-1), 0.5$. Thus:

```

]w = .05, (1 #~ n-1), 0.5
0.5 1 1 1 1 1 1 1 1 0.5
+ / h * w * cube h * i . n+1
4.04

```

The trapezoids provide, in effect, linear approximations to the function between grid points; much better approximations to the integral can be obtained by using groups of $1+2^*k$ points, each group being fitted by a polynomial of degree 2^*k . For example, the case $k = .1$ provides fitting by a polynomial of degree 2 (a parabola) and a consequent weighting of $3\% \sim 1 \ 4 \ 1$ for the three points. If the function to be fitted is itself a polynomial of degree two or less, the integration produced is exact. For example:

```

w = .3%~1 4 1
h = (x = .5) % (n = .2)
]grid = h * i . n+1 [ . f = . ^&2
0 2.5 5
w * f grid
0 8.33333 8.33333
+ / h * w * f grid      Exact integral of ^&2
41.6667
+ / h * w * ^&4 grid      Exact result is 625
651.042

```

Better approximations are provided by several groups of three points, resulting in weights of the form $3\% \sim 1 \ 4 \ 2 \ 4 \ 2 \ 4 \ 2 \ 4 \ 1$. For example, using g groups of $1+2^*k$ points each:

```

n = (g = .4) * 2 * (k = .1)
]h = n %~ x = .5
0.625
]grid = h * i . n+1
0 0.625 1.25 1.875 2.5 3.125 3.75 4.375 5

```

```

1, (4 2 $~ <: 2*g), 1
1 4 2 4 2 4 2 4 1
w=. 3%~ 1, (4 2 $~ <: 2*g), 1
+/h*w**&2 grid      +/h*w**&4 grid
41.6667              625.102

```

This case of fitting by parabolas ($k=. 1$) is commonly used for approximate integration, and is called *Simpson's Rule*. The weights $3\%~1\ 4\ 1$ used in Simpson's rule will now be derived by a general method that applies equally for higher values of k , that is, for any odd number of points. Elementary algebra can be used to determine the coefficients c of a polynomial of degree 2 that passes through any three points on the graph of a function f . The integral of this polynomial (that is, $(0, c\%1\ 2\ 3)\&p$) can be used to determine the exact area under the parabola, and therefore the approximate area under the graph of f .

The appropriate weights are given by the function w , whose definition is presented below, after some examples of its use:

```

W 1
0.333333 1.33333 0.333333
W 2
0.311111 1.42222 0.533333 1.42222 0.311111
3*W 1      45*W 2
1 4 1      14 64 24 64 14
140* W 3
41 216 27 272 27 216 41

```

The derivation of the definition of w is sketched below, together with commentary:

$vm=. \wedge/\sim @i=. i.@>:@+:$	Vandermonde of $i. n$
$vm\ 2$	(odd number of elements)
<pre> 1 0 0 0 0 1 1 1 1 1 1 2 4 8 16 1 3 9 27 81 1 4 16 64 256 </pre>	
$\%.\ vm\ 2$	Inverse of Vandermonde
<pre> 1 0 0 0 0 _2.08333 4 3 1.33333 0.25 _1.45833 _4.33333 4.75 _2.33333 0.458333 _0.416667 1.5 2 1.16667 0.25 0.0416667 _0.166667 0.25 _0.166667 0.0416667 </pre>	

```

integral=. 0:., %.@vm % >:@i      Columns are integrals of
integral 2                          columns of inverse Vm
      0          0          0          0          0
      1          0          0          0          0
      1.04167      2      1.5      0.666667      0.125
      0.486111      1.44444 1.58333      0.777778      0.152778
      0.104167      0.375      0.5      0.291667      0.0625
0.00833333      0.0333333      0.05      0.0333333 0.00833333

```

```
p=. #.~&|. "1 0
```

```
W=. |:integral p +:      Polynomial at double argument
```

```

3*W 1                      45*W 2
1 4 1                      14 64 24 64 14

```

The results produced by `w` may be compared with those derived in more conventional notation, as in Hildebrand [8], page 60 ff. The function `w` can also be defined directly as follows:

```
W=.+: (#. |. @) "0 1 |: @ (0:., %.@(^/~@i) %>:@i=. i.@>:@+:)
```

We now define a function `EW` for extended weights, such that `g EW k` yields the weights for `g` groups of fits for $1+2*k$ points:

```

ew=.+/@(-@(<:@(#@)) * i.@ |. "0 1) { .~>:@([*<:@(#@)))
EW=. ([ ew W@) f.

```

```

2 EW 1
0.333333 1.33333 0.666667 1.33333 0.333333

```

```

3*2 EW 1                      45*2 EW 2
1 4 2 4 1                      14 64 24 64 28 64 24 64 14

```

Finally, we define a conjunction `ai` such that `w ai f x` gives the approximate integral of the function `f` to the point `x`, using the weights `w`:

```

ai=. '+/@(x.&space * x.&[ * y.@(x.&grid)) "0' : 2
grid=. space * i.@#@[
space=. ] % <:@#@[

```

```

3*w=. 1 EW 1
1 4 1
w ai *: x=. 1 2 3 4
0.333333 2.66667 9 21.3333
(x^3)%3
0.333333 2.66667 9 21.3333
(1 EW 2) ai (^&4) x
0.2 6.4 48.6 204.8

```

Weights for Simpson's rule (gives exact results for the square function)

Weights give exact results for integral of fourth power

```
(x^5)%5
0.2 6.4 48.6 204.8
(cir=. 0&o.) 0 0.5 1
1 0.866025 0
```

0&o. is %:0(1: - *:)
cir is altitude of a circle
of unit radius.

```
(2 EW 2) ai cir 1
0.780924
```

Approximation to area under
cir (area of quadrant)

```
4 * (2 EW 2) ai cir 1
3.1237
4*(20 EW 3) ai (0&o.) 1
3.14132
o.1
3.14159
```

Approximation to pi

For use in exercises and in the treatment of interpretations in Section K of Chapter 7, we will define the adverb **I** in terms of the weights **4 EW 5**, that is, four groups of a polynomial approximation of order ten:

```
I=. (4 EW 5) ai
^&9 I x=. 1 2 3 4
0.1 102.4 5904.9 104858
(x^10) % 10
0.1 102.4 5904.9 104858
```

C1 Use the integral to determine the area under the square root function up to various points.

C2 Since the graphs of the square and the square root intersect at 0 and 1, they enclose an area. Determine its size.

[(%:I-*:I) 1 or (%:-*:) I 1]

C3 Experiment with the expression $(f - f \text{ I } D) \ x$ for various functions f and arguments x .

D. DIFFERENTIAL EQUATIONS

A function may be defined directly, as in $f=.^{\wedge}3$ or $g=.0 \ 0 \ 0 \ 1&p$, where $p=. \ \#.\sim\&| \ . \ " \ 1 \ 0$ is the polynomial function. It may also be defined indirectly by an equation that specifies some relation that it must satisfy. For example (using $D=. \ ("0) (D.1) \)$:

1. q is the inverse of the cube.

A function that satisfies this equation may be expressed directly in various ways. For example:

```
q=. ^&3
cube=. . ^&3
cube x=. 0 1 2 3 4
0 1 8 27 64
q cube x
0 1 2 3 4
cube q x
0 1 2 3 4
q=. cube (INV=. ^:_1)
q cube x
0 1 2 3 4
```

Inverse adverb

2. x is the reciprocal of the cube.

```
r=. cube (R=. %0)
r x
_ 1 0.125 0.037037 0.015625
(r * cube) x
0 1 1 1 1
```

Reciprocal adverb

3. The derivative of s is the cube.

```
s=. ^&4 % 4:
s D x
0 1 8 27 64
cube x
0 1 8 27 64
s x
0 0.25 4 20.25 64
```

A stated relation may not specify a function completely. For example, the equation for Example 3 is also satisfied by the alternative function $as=. 8:+s$. Thus:

```
(as=. 8:+s) x
8 8.25 12 28.25 72
as D x
0 1 8 27 64
```

Further conditions may therefore be stipulated to define the function completely. For example, if it is further required that $s\ 2$ must be 7, then s is completely defined. Thus:

```
s=. 3: + s f.
s 2
7
s x
3 3.25 7 23.25 67
s D x
0 1 8 27 64
```

An equation that involves derivatives of the function being defined is called a *differential* equation. The remainder of this chapter will

use simple differential equations to define an important collection of functions, including the *exponential*, *hyperbolic*, and *circular* (or *trigonometric*).

We will approach the solution of differential equations through the use of polynomials. Because a polynomial includes one more monomial (or *term*) than its derivative, it can never exactly equal the derivative, and we consider functions that approximate the desired solution. However, for the cases considered, successive coefficients decrease rapidly in magnitude, and approximation can be made as close as desired. Consideration of the convergence of such approximations is deferred to Chapter 8.

D1 Experiment with the expressions of this section.

E. GROWTH $f' = f$

If the derivative of a function is equal to (or proportional to) the function itself, it is said to grow *exponentially*. Examples of exponential growth include continuous compound interest, and the growth of a well-fed colony of bacteria.

If f is the polynomial $c \cdot x^p$, then the derivative of f is the polynomial with coefficients $d \cdot c$, where $d = \frac{p}{i} \cdot c$. Thus:

```

]c=. 1, (1*1), (1*1*2), (1*1*2*3), (1*1*4), (%15), (%16)
1 1 0.5 0.166667 0.0416667 0.00833333 0.00138889
  c*i.#c
0 1 1 0.5 0.166667 0.0416667 0.00833333
  ).c*i.#c
1 1 0.5 0.166667 0.0416667 0.00833333
  d c
1 1 0.5 0.166667 0.0416667 0.00833333

```

In this case the coefficients of the derivative polynomial agree with the original coefficients except for the missing final element. The same is true for any coefficients produced by the following *exponential coefficients* function:

```

ec=. %0!
]c=. ec i.7
1 1 0.5 0.166667 0.0416667 0.00833333 0.00138889
  d c
1 1 0.5 0.166667 0.0416667 0.00833333

```

Consequently, the function $c\&p$ is approximately equal to its derivative. For example:

```
c&p x=. 0 1 2 3
1 2.71806 7.35556 19.4125
(d c)&p x
1 2.71667 7.26667 18.4
```

The primitive *exponential* function, denoted by $\wedge$, is defined as the limiting case for large n . For example:

```
12.8 ": ^ x
1.00000000 2.71828183 7.38905610 20.08553692
12.8 ": (ec i.20) p x
1.00000000 2.71828183 7.38905610 20.08553692
```

The related function $\wedge\theta(x\&*)$ grows at a rate *proportional* to the function, the ratio being x . For example:

```
x=.0.1
q=. ^\theta(x&*)
q D x
0.1 0.110517 0.12214 0.134986
x * q x
0.1 0.110517 0.12214 0.134986
```

This relation follows from theorems of Table B1 as follows:

$f = . \wedge$	
$g = . x\&*$	
$q = . f\theta g$	
$\wedge\theta(x\&*) D$	
$q D$	Definition of q
$f\theta g D$	Definition of q
$(f D\theta g * g D)$	$\theta 5$
$(f\theta g * g D)$	Definition of f and of $\wedge$ as $\wedge D$
$(f\theta g * x^0)$	$\theta 2$
$(q * x^0)$	Definition of q
$(x^0 * q)$	Times is commutative
$(x^0 * \wedge\theta(x\&*))$	Definition of q

This sequence of equivalent expressions establishes the fact that the leading expression $\wedge\theta(x\&*) D$ is equivalent to the last expression

$x^0 * ^0(x)$. Finally, $(x^0 * ^0(x)) x$ is equivalent to the expression $x * q x$ used in the examples above.

It is also interesting to compare the exponential coefficients used above with those obtained by applying the adverb **FIT** (of §1E) to the primitive exponential:

```
FIT=. 'y. %. ^/6(i. x.)' : 2
```

```
7(.10 FIT ^ 10%~i.10
```

```
1 1 0.5 0.166667 0.0416658 0.00833607 0.00138336
```

```
ec i.7
```

```
1 1 0.5 0.166667 0.0416667 0.00833333 0.00138889
```

E1 Test the proof presented in this section by applying each expression in it to the argument $x=. i. 6$.

F. DECAY $f D = -^0 f$

A function whose derivative is equal to or proportional to its negation is *decaying* at a rate proportional to itself. Interpretations include the charge of water remaining in a can punctured at the bottom, and the electrical charge remaining in a capacitor draining through a resistor; the rate of flow (and therefore of loss) is proportional to the pressure provided by the remaining charge at any time.

The coefficients of a polynomial defining such a function must be similar to that for growth, except that the elements must alternate in sign. Thus:

```
eca=. _1% ^ * ec
```

```
eca i.7
```

```
1 _1 0.5 _0.166667 0.0416667 _0.00833333 0.00138889
```

```
_d eca 7
```

```
_1 1 _0.5 0.166667 _0.0416667 0.00833333
```

```
(eca 20) %p x
```

```
1 0.367879 0.135335 0.0497871 0.0183153
```

```
(d eca 20) %p x
```

```
_1 _0.367879 _0.135335 _0.0497871 _0.0183175
```

The relation between the growth and decay functions will be explored in exercises and in Chapter 6.

F1 Define a function **pp** such that $(a \text{ pp } b) \% p$ is equivalent to the product $(a \% p * b \% p)$ and test it for the case $a=.1 \ 2 \ 1$ [$b=.1 \ 3 \ 3 \ 1$.

```
[ pp=. +//.0(*) ]
```

F2 Predict the value of the first few elements of `(ec pp eca) i.7` and enter the expression to validate your prediction.

G. HYPERBOLICS $f D D = f$

As illustrated in the interpretations in Section 1 D, the second derivative of a function may be construed as its *acceleration*. Many phenomena are described by functions defined in terms of their acceleration.

We will again use polynomials to approximate functions, first a function that is equal to its second derivative. Since the *second* derivative of the exponential $\wedge$ is also equal to itself, the coefficients `ec i.n` would suffice. However, we seek new functions and therefore add the restriction that $f D$ must *not* equal f .

Coefficients satisfying these requirements can be obtained by suppressing (that is, replacing by zeros) alternate elements of `ec i.n`. Thus:

```
2|i.n=. 9
0 1 0 1 0 1 0 1 0
hsc=. 2&| * ec
|c=. hsc i.n
0 1 0 0.166667 0 0.00833333 0 0.000198413 0
d c
1 0 0.5 0 0.0416667 0 0.00138889 0
d d c
0 1 0 0.166667 0 0.00833333 0
```

The result of `d c` was shown above to make clear that the *first* derivative differs from the function. However, it should also be apparent that *it* qualifies as a second function that equals its second derivative. We therefore define a corresponding function `hcc` as follows:

```
hcc=. 0&=@(2&|) * ec
hcc i.n
1 0 0.5 0 0.0416667 0 0.00138889 0 2.48016e_5
d d hcc i.n
1 0 0.5 0 0.0416667 0 0.00138889
```

The limiting values of the corresponding polynomials `(hsc n)&p` and `(hcc n)&p` are called the *hyperbolic sine* and *hyperbolic co-*

sine, respectively. They are the functions defined by `hsin=. 5&o.` and `hcos=. 6&o.`. Thus:

```
(hsc i.20)&p x=. 0 1 2 3 4
0 1.1752 3.62686 10.0179 27.2899
  hsin x
0 1.1752 3.62686 10.0179 27.2899
  (hcc i.20)&p x
1 1.54308 3.7622 10.0677 27.3082
  hcos x
1 1.54308 3.7622 10.0677 27.3082
```

It should also be noted that each of the hyperbolic functions is the derivative of the other. Further properties of these functions will be explored in Chapter 6. In particular, it will be seen that a plot of one against the other yields a hyperbola. The more pronounceable abbreviations *cosh* and *sinh* (pronounced *cinch*) are also used for these functions.

G1 Enter the definition of the polynomial product `pp` of Exercise F1; use it to evaluate the expression `(hcc -&(pp~) hsc) i.10.` Comment on the results.

G2 Enter expressions to compare the polynomial approximations to the hyperbolic functions with the functions `5&o.` and `6&o.`.

G3 Predict the results of the expression `(6&o. -&*: 5&o.) i. 9.`

H. CIRCULARS $f'' = -f$

It may be noted that the hyperbolics, like the exponential, continue to grow with increasing arguments. This is not surprising, since their acceleration increases with the increase of the function.

We now consider functions whose acceleration is *opposite* in sign to the functions themselves, a characteristic that leads to *periodic* functions, whose values repeat as arguments grow. These functions are useful in describing periodic phenomena such as the oscillations in a mechanical system (the motion of a weight suspended on a spring) or in an electrical system (a coil connected to a capacitor).

Appropriate polynomial coefficients are easily obtained by alternating the signs of the non-zero elements resulting from `hsc` and `hcc`.

Thus:

```

sc=. _1&^@(3&=)@(4&|) * hsc
cc=. _1&^@(2&=)@(4&|) * hcc
sc i.n
0 1 0 _0.166667 0 0.00833333 0 _0.000198413 0
cc i.n
1 0 _0.5 0 0.0416667 0 _0.00138889 0 2.48016e_5
(sc i.20)&p x
0 0.841471 0.909297 0.14112 _0.756803
sin x
0 0.841471 0.909297 0.14112 _0.756802
(cc i.20)&p x
1 0.540302 _0.416147 _0.989992 _0.653644
cos x
1 0.540302 _0.416147 _0.989992 _0.653644

```

It may be surprising that these functions defined only in terms of their derivatives are precisely the *sine* and *cosine* functions of trigonometry (expressed in terms of arguments in *radians* rather than *degrees*); these relations are examined in Section 6F.

H1 Repeat Exercises G1 and G2 with modifications appropriate to the circular functions.

[sc +&(pp~) sc)]

I. SCALING

The function $\wedge @ (x \& *)$ used in Section B is an example of *scaling*; its argument is first multiplied by the *scale factor* x before applying the main function $\wedge$. Such scaling is generally useful, and we define a more convenient conjunction for the purpose as follows:

AM=. [. @ (&*) Atop Multiplication

For example:

```

&(0.1&*) x=. 0 1 2 3 4
1 1.10517 1.2214 1.34986 1.49182
^ AM 0.1 x
1 1.10517 1.2214 1.34986 1.49182

```

Thus, **f AM x** may be read as “**f** atop multiplication (by) **x**”. Also:

```

^ AM 0.1 D x
0.1 0.110517 0.12214 0.134986 0.149182
0.1 * ^ AM 0.1 x
0.1 0.110517 0.12214 0.134986 0.149182

```

J. ARGUMENT TRANSFORMATIONS.

Scaling is only one of many useful argument transformations, and we define two further conjunctions, *atop addition* and *atop polynomial*:

```

AA=. [. @ (&+)
AP=. [. @ (&p)

```

In Section H it was remarked that the circular functions *sin* and *cos* "repeat" their values after a certain period. Thus:

```

per=. 6.28
cos x
1 0.540302 _0.416147 _0.989992 _0.653644
cos AA per x
0.999995 0.54298 _0.413248 _0.989538 _0.656051

```

Experimentation with different values of *per* can be used to determine a better approximation to the true period of the cosine.

The conjunction *AP* provides a more general transformation. Thus:

```

f AA 3 AM 4 is f AP 3 4
f AM 3 AA 4 is f AP 12 3

```

A function *FfC* to yield Fahrenheit from Celsius can be used to further illustrate the use of argument transformation:

```

FfC=. 32&+@ (1.8&*)
fahr=. _40 0 100

FfC fahr
_40 32 212

] AA 32 AM 1.8 fahr
_40 32 212

] AP 32 1.8 fahr
_40 32 212

```

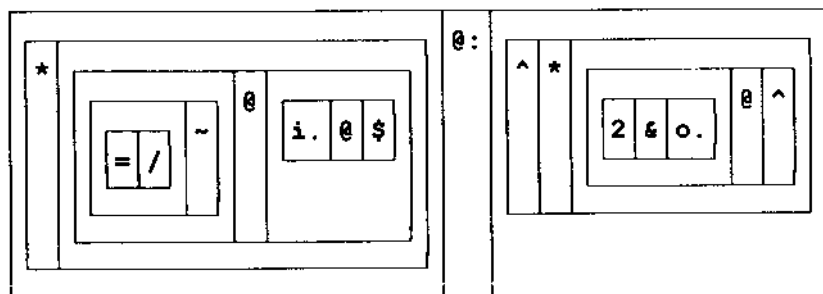
The following derivatives are easily obtained by substitution and the use of Table B1:

Function	Derivative
f AA x	f D AA x
f AM x	(f D AM x * x"0)
f AP c	(f D AP c * (d c)&p)

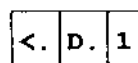
K. D. REVISITED

As stated in §1C, the expression $\mathcal{E} \text{ D. } n$ yields the n th derivative of the function $\mathcal{E}$. If the derivative of $\mathcal{E}$ can be obtained formally from the rules of Table B1 and from the function definitions in later sections, this is done; otherwise it is obtained by approximation. For example, the formal derivative of $1\&0.$ (sine) is $2\&0.$, but the derivative of the integer part function $\<.$ is approximated. Thus:

$$1\&0.\theta^{\text{D. } 1}$$



$$\<.\text{D. } 1$$



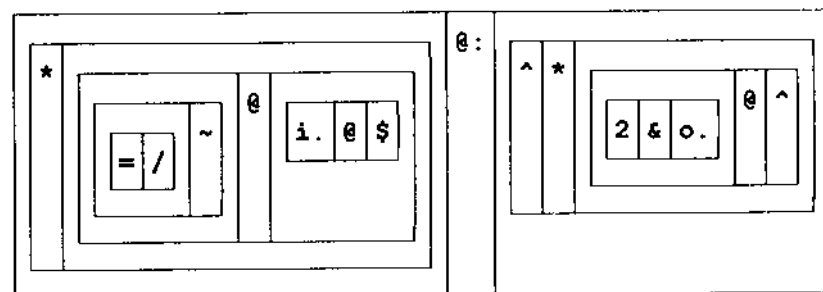
Moreover, the derivative of a named function is approximated (because its referent might be re-assigned), but the named function can be fixed by the adverb $\mathcal{F}.$. For example:

$$\text{sinexp} = 1\&0.\theta^{\text{D. } 1}$$

$$\text{sinexp D. } 1$$



$$\text{sinexp f. D. } 1$$



Chapter 3: VECTOR CALCULUS

A. INTRODUCTION

The discussion of the functions $\nabla_0 = \cdot$ and $\nabla_1 = \cdot$ in Section C of Chapter 1 introduced the essential notion of the vector derivative of a function f : it is a function that yields the (scalar) derivative of each atom of the result of f with respect to each atom of the argument of f .

For example, the function ∇_0 produces a rank-0 (scalar, or atomic) result from a rank-1 (vector) argument, and is therefore said to have *form 0 1* or to *be a 0 1 function*; its derivative produces a rank-1 result from a rank-1 argument, and has *form 1 1*. Similarly, the function ∇_1 has *form 1 1*, and *its derivative* has *form 2 1*. In general, the derivative of a function of *form r, a* has *form $(r+a), a$* .

Strictly speaking, vector calculus concerns only functions of the forms *0 1* and *1 1*; other forms tend to be referred to as *tensor analysis*. Since the analysis remains the same for other forms, we will not restrict attention to the forms *0 1* and *1 1*. However, we will normally restrict attention to three-space (as in $\nabla_0 \cdot \mathbf{a}$ for the volume of a box) or two-space (as in $\nabla_1 \cdot \mathbf{a}$ for the area of a rectangle), although an arbitrary number of elements may be treated.

We will adopt the convention that a name ending in the digits r and a denotes an r, a function. For example, ∇_0 is a scalar function of a vector, ∇_1 is a vector function of a vector, and ∇_2 is a scalar function of a matrix (such as the determinant $\det = \cdot$).

We will call the *form $(r+a), a$* vector derivative resulting from a *form r, a* function its *complete derivative* or *gradient*, to distinguish it from other important derivatives that are based upon it. These are commonly simpler in the sense that they produce results of rank lower than the gradient. For example, the gradient of a *1 1* function ∇_1 is a *2 1* function, the determinant of which is a *0 1* function called the *Jacobian* of ∇_1 , and the sum of the diagonal elements of which is a *0 1* function called the *divergence* of ∇_1 .

B. GRADIENT

The fundamental vector derivative adverb is the gradient, defined by:

$$\text{GRAD} = \cdot \quad (D.1) \quad \text{See §1C}$$

We will illustrate its application to a number of functions:

```

E01=. +/@:":1
F01=. %:@E01
G01=. 4p1"1 * *: @F01      Four pi times square of F01
H01=. %G01

p=. 1 2 3
E01 p      F01 p      G01 p      H01 p
14      3.74166      175.929      0.00568411
H01 i. 3 3
0.0159155 0.00159155 0.000534077
E01 GRAD p
2 4 6
F01 GRAD p
0.267261 0.534522 0.801784
G01 GRAD p
25.1327 50.2655 75.3982
H01 GRAD p
_0.000812015 _0.00162403 _0.00243604

```

B1 Develop interpretations for each of the functions defined above.

[$E01\ p$ is the square of the distance (from the origin) to a point p .

$F01\ p$ is the distance to a point p , or the radius of the sphere (with centre at the origin) through the point p .

$G01\ p$ is the surface area of the sphere through the point p .

$H01$ is the intensity of illumination at point p provided by a unit light source at the origin.]

B2 Provide direct definitions (that is, without using the derivative adverb) of functions equivalent to the derivatives of each of the functions defined above.

```

[ E11=. +:":1
F11=. -: @% @% : @E01 * E11
G11=. 4p1% * @E11
H11=. -@% @% : @G01 * G11 ]

```

C. AUXILIARY ADVERBS

We will first develop a set of adverbs that do not themselves involve derivatives, but will be used in the definition of further forms of derivatives. Thus:

```

DET=. -/ . * @
TRACE=. ((<0 1)&|:@) (+/@)
NORM=. norm@
  norm=. +/^: (]` (#@$)` (* CSK@#)) % !@ (#-#@$)
  CSK=. par@indices=. (] #: i.)@ (#~)
  par=. _1&^@ (+/)@ (>:@ (#>)@C.) :: 0:"1
f=. ^/~"1
f a=. 1 2 3
1 1 1
2 4 8
3 9 27
  f DET a
12
  f DIAG a
1 4 27
  f TRACE a
32

```

C1 Experiment with the auxiliary adverbs on $f=. \wedge/\sim"1$ and other 2 1 functions.

D. JACOBIAN

We will now define the Jacobian, and illustrate its use on a 1 1 function:

```

JAC=. GRAD DET
VOLS=. */\ "1
VOLS a
1 2 6
  VOLS JAC a
2

```

The Jacobian may be interpreted as the *volume derivative*, or rate of change of volume produced by application of a function. This interpretation is most easily appreciated in the case of a linear function. We will begin with a linear function in 2-space, in which case the "volume" of a body is actually the area:

```

mp=. +/ . *
]m=. 2 2$2 0 0 3
2 0
0 3

```

```

L11=. mp&m"1
]fig1=.>1 1;1 0;0 0;0 1
1 1
1 0
0 0
0 1
]fig2=. L11 fig1
2 3
2 0
0 0
0 3
L11 JAC 1 1
6
L11 JAC 1 0
6
L11 JAC fig1
6 6 6 6

```

It should be noted that the result of the Jacobian is indeed the ratio of the areas of **fig1** and **fig2**. Moreover, for a linear function, the value of the Jacobian is the same at every point.

D1 Provide an interpretation for the function $\kappa_{11} = (H_{11} * J)^{-1}$.

[The result of κ_{11} is the direction and magnitude of the repulsion of a negative electrical charge from a positive charge at the origin. The function $-\kappa_{11}$ may be interpreted as gravitational attraction.]

D2 What is the relation between the Jacobian of the linear function **L11** and the determinant of the matrix **m** used in its definition?

D3 What is the relation between the Jacobians of two linear functions **LA11** and **LB11** and the Jacobian of **LC11** = **LA11@LB11** (their composition).

[**TEST** = **LA11@LB11 JAC** |@- **LA11 JAC** * **LB11 JAC**]

D4 Define functions **LA11** and **LB11**, and test the comparison expressed in the solution to Exercise 3 by applying **TEST** to appropriate arguments.

D5 The Jacobian of the linear **LR11** = **mp&(>0 1;1 0)"1** is **_1**. State the significance of a negative Jacobian.

[Plot figures **fig1** and **fig2**, and note that one can be moved smoothly onto the other "without crossing lines". Verify that this

cannot be done with `fig1` and `LR11 fig1`; it is necessary to "lift the figure out of the plane and flip it over". A transformation whose Jacobian is negative is said to involve a "reflection".]

D6 Enter, experiment with, and comment upon the following functions:

```
RM2=. 2 2&$@ (1 1 1 1&*) @ (2 1 1 2&o.) "0
R2=. ( ) mp RM2@ ( ) "0 1
```

[`R2` is a linear function that produces a *rotation in 2-space*; the expression `a R2 fig` rotates a figure (such as `fig1` or `fig2`) about the origin through an angle of `a` radians in a counter-clockwise sense, without deforming the figure.]

D7 What is the value of the Jacobian of a rotation `a&R2"1`?

D8 Enter an expression to define `FIG1` as an 8 by 3 table representing a cube, making sure that successive coordinates are adjacent, for example, 0 1 1 must not succeed 1 1 0. Define 3-space linear functions to apply to `FIG1`, and use them together with `K11` to repeat Exercises 1-5 in 3-space.

D9 Enter, experiment with, and comment upon the functions

```
RM3=. 1 0 0&, @ (0&, .) @ RM2
R30=. ( ) mp RM3@ ( ) "0 1
```

[`a&R30"1` produces a rotation through an angle `a` in the plane of the last two axes in 3-space (or *about* axis 0). Test the value of the Jacobian.]

D10 Define functions `R31` and `R32` that rotate about the other axes, and experiment with functions such as `a1&R31@ (a2&R30) "1`.

[Experiment with the permutations `p=. 2&A.` and `p=. 5&A.` in the expression `p&. | : @ p @ RM3 o. %2`, and use the ideas in functions defined in terms of `R30`.]

E. DIVERGENCE AND LAPLACIAN

The *divergence* and *Laplacian* are defined and used as follows:

```
DIV=. GRAD TRACE
LAP=. GRAD DIV
f=. +/\ "1
f a
```

1 3 6

f GRAD a	f DIV a
1 0 0	3
1 1 0	
1 1 1	
g=. +/@{} ^ >:@i.@#)"1	
g a	g LAP a
32	20

It is difficult to provide a helpful interpretation of the divergence except in the context of an already-familiar physical application, and the reader may be best advised to seek interpretations in some familiar field. However, in his *Advanced Calculus*[9], F.S. Woods offers the following:

"The reason for the choice of the name *divergence* may be seen by interpreting $\mathbf{F}$ as equal to $\rho\mathbf{v}$, where ρ is the density of a fluid and $\mathbf{v}$ is its velocity. ... Applied to an infinitesimal volume it appears that $\text{div } \mathbf{F}$ represents the amount of fluid per unit time which streams or diverges from a point."

F. CURL

The gradient of a 1 1 function $\mathbf{f}$ is a 2 1 function, and therefore produces a matrix. The normal to this matrix is called the *curl* of $\mathbf{f}$:

```
CURL=. GRAD NORM
f=. +/\
f CURL x
_1 1 _1
```

Interpretation of the curl is perhaps even more intractable than the divergence. Again Woods offers some help:

"The reason for the use of the word *curl* is hard to give without extended treatment of the subject of fluid motion. The student may obtain some help by noticing that if $\mathbf{F}$ is the velocity of a liquid, then for velocity in what we have called irrotational motion, $\text{curl } \mathbf{F}=0$, and for vortex motion, $\text{curl } \mathbf{F}\neq 0$.

It may be shown that if a spherical particle of fluid be considered, its motion in a time dt may be analyzed into a translation, a deformation, and a rotation about an instantaneous axis. The curl of the vector $\mathbf{v}$ can be shown to have the direction of this axis and a magnitude equal to twice the instantaneous angular velocity."

In his *Div, Grad, Curl, and all that* [10], H.M. Schey makes an interesting attempt to introduce the concepts of the vector calculus in terms of a single topic. His first chapter begins with:

In this text the subject of the vector calculus is presented in the context of simple electrostatics. We follow this procedure for two reasons. First, much of vector calculus was invented for use in electromagnetic theory and is ideally suited to it. This presentation will therefore show what vector calculus is, and at the same time give you an idea of what it's for. Second, we have a deep-seated conviction that mathematics—in any case *some* mathematics—is best discussed in a context which is not exclusively mathematical.

Schey's treatment includes Maxwell's equations which, in Heaviside's elegant formulation, exhibit the powers of div, grad, and curl in joint use.

F1 Experiment with **GRAD**, **CURL**, **DIV**, and **JAC** on the functions in Exercise B2.

F2 Experiment with **GRAD**, **CURL**, **DIV**, and **JAC** on the following 1 1 functions:

```
q=. *:               r=. 4%a. @ q
s=. 1 1 _1%* @ r
t=. 3%a. @ ^ @ -
u=. ]% (+/@(*~)) ^ 3x2"0
```

F3 Enter the definitions $x=. 0\%$, $y=. 1\%$, $z=. 2\%$, and use them to define the functions of the preceding exercise in a more conventional form.

```
[ as=. *: @ z, *: @ x, - @ *: @ y
  at=. ^ @ - @ y, ^ @ - @ z, ^ @ - @ x
  au=. (x, y, z) % (*: @ x + *: @ y + *: @ z) ^ 3x2"0 ]
```

F4 Experiment with **LAP** on various 0 1 functions.

F5 Express the cross product of §6G so as to show its relation to **CURL**. See §6I for help.

```
[ CR=. */ NORM      CURL=. GRAD NORM ]
```


Chapter 4: DIFFERENCE CALCULUS

A. INTRODUCTION

Although published some fifty years ago, Jordan's *Calculus of Finite Differences* [11] still provides an interesting treatment. In his introductory section on *Historical and Biographical Notes*, he contrasts the difference and differential (or *infinitesimal*) calculus as follows:

Two sorts of functions are to be distinguished. First, functions in which the variable x may take every possible value in a given interval; that is, the variable is continuous. These functions belong to the domain of the *Infinitesimal Calculus*. Secondly, functions in which the variable takes only the given values $x_0, x_1, x_2, \dots, x_n$; then the variable is discontinuous. To such functions the methods of Infinitesimal Calculus are not applicable. The *Calculus of Finite Differences* deals especially with such functions, but it may be applied to both categories.

The present brief treatment is restricted to three main ideas:

- 1) The development of a family of functions which behaves as simply under the *difference* (secant slope) adverb as does the family of power functions x^n under the derivative adverb.
- 2) The definition of a polynomial function in terms of this family of functions.
- 3) The development of a linear transformation from the coefficients of such a polynomial to the coefficients of an equivalent ordinary polynomial.

We will also restrict attention to scalar functions, using the scalar derivative $D = \frac{dy}{dx}$ (D.1); and will first review the discussion of secant slopes in Section B of Chapter 1, and the results derived in Exercises B2 to B4 of that chapter:

$S = \frac{y_2 - y_1}{x_2 - x_1}$: 2 Slope conjunction

$r = \frac{y_2 - y_1}{x_2 - x_1}$: 5

$r = \frac{y_2 - y_1}{x_2 - x_1}$

r -slope of cube

2.71 11.41 26.11 46.81 73.51

$(3x^2) + (3x) + (x^2)$

2.71 11.41 26.11 46.81 73.51

```

p=. i.0] ^~/ [
x p 4
1 1 1 1 1
1 2 3 4 5
1 4 9 16 25
1 8 27 64 125
x p -4
0.001 0.01 0.1 1
+/_ 1 4 _6 4 *(x p 4) * (x p -4)
3.439 29.679 102.719 246.559 485.199
x S (^&4) x
3.439 29.679 102.719 246.559 485.199

```

As illustrated above, the slope of the power function x^n can be written as a sum of n terms. For the limiting case of the derivative ($x=0$) this result simplifies to the leading term, which can also be written (in terms of another power function) as simply $n \cdot x^{n-1}$. Consequently, the derivative of the polynomial $c \cdot x^p$ can be written in terms of another polynomial as $(\sum c \cdot i \cdot x^{i-1}) \cdot x^p$.

This is an important property of the family of power functions, and we seek another family of functions that behaves similarly under the x -slope. We begin by adopting the names p_0 and p_1 and p_2 , etc., for the functions x^0 and x^1 and x^2 , and by showing how each member of the family can be defined in terms of another. Thus:

```
p4=. ]*p3=. ]*p2=. ]*p1=. ]*p0=. 1:"0
```

We now recall (from §1B) expressions for the slopes and derivatives of the sum and product of functions (using the adverb $xS=. x \cdot S$ for brevity):

Function:	$f + g$	$f \cdot g$
Slope:	$f \cdot xS + g \cdot xS$	$(f \cdot g \cdot xS) + (f \cdot xS \cdot g) - (x^0 \cdot f \cdot xS \cdot g \cdot xS)$
Deriv:	$f \cdot D + g \cdot D$	$(f \cdot g \cdot D) + (f \cdot D \cdot g)$

Since the derivative of the identity function $]$ is $1:$, expressions for the derivatives of the power functions can be derived using the expressions for the sum and product in informal proofs as follows:

$p_0 \cdot D$	$p_1 \cdot D$	$p_2 \cdot D$
$1: \cdot D$	$(] \cdot p_0) \cdot D$	$(] \cdot p_1) \cdot D$
$0:"0$	$(] \cdot p_0 \cdot D) + (] \cdot D \cdot 1:)$	$(] \cdot p_1 \cdot D) + (] \cdot D \cdot p_1)$
	$(] \cdot 0:.) + (1: \cdot 1:.)$	$(] \cdot 1:.) + (1: \cdot p_1)$
	$1:"0$	$p_1 + p_1$
		$2: \cdot p_1$

p3 D	p4 D
([*p2]D	([*p3]D
([*p2 D]+([D*p2)	([*p3 D]+([D*p3)
([*2:*p1]+(1:*p2)	([*3:*p2]+(1:*p3)
(2:*p2)+p2	(3:*p3)+p3
3:*p2	4:*p3

Each expression in the proofs may be tested by applying it to an argument such as $x = . 1 . 6$, first enclosing any fork in parentheses.

Beginning with the function $q0 = . 1 : "0$ and the fact that the slope of the identity function is also $1 :$, we will define functions $q1$, $q2$, $q3$, etc., such that the slope $q1 \text{ } rS$ is $1 : *q0$, the slope $q2 \text{ } rS$ is $2 : *q1$, and so on. In proofs we will use $R = . r "0$ for clarity.

Defining $q1 = .] * q0 = . 1 : "0$ gives the desired result because:

```

q1 rS
([ * q0) rS
([ * q0 rS) + ([ rS * q0) - (R * ] rS * q0 rS)
([ * 0 :) + (1: * q0) - (R * 1: * 0 :)
1: * q0
q0

```

however, the corresponding definition $q2 = .] * q1$ fails to satisfy because:

```

q2 rS
([ * q1) rS
([ * q1 rS) + ([ rS * q1) - (R * 1: * q0)
([ * q0) + (1: * q1) - (R * 1: * q0)
q1 + q1 - (R * q0)
(2: * q1) - (R * q0)

```

Since the slope of the constant function R is $0 :$, it follows that $(R * q1) \text{ } rS$ is $R * q0$, or simply R ; addition of $R * q1$ to the definition of $q2$ would therefore remove the unwanted term in the slope. Thus if $q2 = . ([+ R) * q1$, then:

```

q2 rS
(([ + R) * q1) rS
([ * q1) rS + (R * q1) rS
([*]) rS + (R*) rS
(([*1:] + (1:*)) - (R*1:*1:)) + (R*) rS

```

```

([+]-R) + (R*) rS
([+]-R) + (R*1:)+(0:*)-(R*1:*0:)
([+]-R) + R
]+]
q1+q1
2: * q1

```

This suggests further definitions of the form:

```
q4=. ([ + 3: * R) * q3=. ([ + 2: * R) * q2
```

and similar analysis will show that they provide the desired r -slopes.

We may therefore define q_4 more directly by saying that $q_4 \ x$ is $*/"1 \ x+/r*i.4$, and more generally by using the expression $*/"1 \ x+/r*i.n$.

A1 Define $\mathfrak{D}$ and p_4 as in the text, and evaluate $p_4 \ \mathfrak{D} \ x=. \ i.6$.

Then continue to test the proof that $p_4 \ \mathfrak{D}$ is equivalent to $4: * p_3$ by entering and evaluating each line of it.

A2 Test the proof that $q_2 \ rS$ is equal to $2: * q_1$.

A3 Prove that $q_3 \ rS$ is equivalent to $3: * q_2$.

B. STOPE FUNCTIONS

The factors that occur in the final expression for the q functions in §A change by steps suggestive of those in a mine stope that follows a sloping vein; $*/"1 \ x+/r*i.n$ will therefore be called an r -stope of order n . Since (for integer values of n) the power expression x^n can also be written in the same form (with $r=. \ 0$), the stope function can be considered as a variant of power; it is written as $^! . r$, using the *fit* conjunction $!$. For example:

```

x=. 0 1 2 3 4 5 [ n=. 4 [ r=. 0.1
st=. ^! . r
x st n
0 1.716 21.252 98.208 296.184 702.78
*/"1 x +/ r* i. n
0 1.716 21.252 98.208 296.184 702.78
qn=. st&n
qn x
0 1.716 21.252 98.208 296.184 702.78

```

```

rS=. r S
qn rS x
0 5.28 36.96 119.04 275.52 530.4
n* st&(n-1) x
0 5.28 36.96 119.04 275.52 530.4

```

Analogous to $\hat{a}_n D$

B1 Enter the expressions of this section, and experiment with them for different values of x and n , including the cases $x=.1$ and $x=._1$.

C. STOPE POLYNOMIALS

A polynomial can be generalized by defining it in terms of a stope function rather than the power function. To this end we adopt the Vandermonde form of the polynomial presented in §1E, and define an adverb **VM** such that $x \text{ VM}$ is a polynomial based on an r -stope:

```

VM=. '[ ^!.x./ i.@#0]' : 1
x=. 0 1 2 3 4 5
c=. 1 3 3 1
r=. 0.1

      x 0 VM c
1 0 0 0
1 1 1 1
1 2 4 8
1 3 9 27
1 4 16 64
1 5 25 125

      x r VM c
1 0 0 0
1 1 1.1 1.32
1 2 4.2 9.24
1 3 9.3 29.76
1 4 16.4 68.88
1 5 25.5 132.6

```

```

mp=. +/ . *      Matrix product
(x 0 VM c) mp c      (x r VM c) mp c
1 8 27 64 125 216      1 8.62 28.84 67.66 131.08 225.1

```

We now define a corresponding *stope polynomial* adverb:

```

SP=. ' ( ) x. VM ( ) +/ . * [' : 1
c 0 SP x
1 8 27 64 125 216
c r SP x
1 8.62 28.84 67.66 131.08 225.1

```

The results of §B can be used to show that the r -slope of an r -polynomial behaves analogously to the derivative of an ordinary polynomial. For example:

```

p=. 0 SP      rp=. r SP

```

```

      c p x
1 8 27 64 125 216
      c rp x
1 8.62 28.84 67.66 131.08 225.1
      D=. ("0) (D.1)
      c&p D x
3 12 27 48 75 108
      derc=. ).@{) * i.@#) The derivative coefficient from §1E
      (derc c) p x
3 12 27 48 75 108
      c&rp rS x
3 12.3 27.6 48.9 76.2 109.5
      (derc c) rp x
3 12.3 27.6 48.9 76.2 109.5

```

Integration of slope polynomials also behaves analogously:

```

      intc=. 0: , ] % >:@i.@# The integral coefficient from §1E
      (intc c)&p x
0 3.75 20 63.75 156 323.75
      (intc c)&p D x
1 8 27 64 125 216
      (intc c)&rp x
0 4.399 22.853 71.262 171.526 351.545
      (intc c)&rp rS x
1 8.62 28.84 67.66 131.08 225.1
      c&rp x
1 8.62 28.84 67.66 131.08 225.1

```

C1 Enter the expressions of this section, and experiment with similar expressions.

D. COEFFICIENT TRANSFORMATIONS

It is important to be able to express an ordinary polynomial as an equivalent slope polynomial, and vice versa. We will therefore define a conjunction **FROM** such that **(a FROM b c) mp c** produces from **c** (the coefficients of a *b*-polynomial) the coefficients of an equivalent *a*-polynomial:

```

FROM=. '(x. VM %. y. VM)~ @i.@#' : 2
mp=. +/ . * Matrix product
]cr=. ((-r) FROM 0 c) mp c
1 2.72 2.7 1

```

```

cr r SP x
1 8 27 64 125 216
c 0 SP x
1 8 27 64 125 216

```

The definition of **FROM** will be examined in exercises. The case **_1 SP** is called the *falling factorial* function, and the corresponding matrix argument for the linear transformation is called the *Stirling numbers of the first kind*. Thus:

```

_1 FROM 0 c
1 0 0 0
0 1 1 2
0 0 1 3
0 0 0 1
0 FROM _1 c      Stirling numbers of the second kind
1 0 0 0
0 1 1 1
0 0 1 3
0 0 0 1

```

D1 Examine the definition of **FROM** in the following steps:

1. Enter `r=.0.1[c=.1 3 3 1[x=.i.4 and ]mr=. r VM x and ]m0=. 0 VM x`.
2. Enter `q=. mr % m0` to obtain the “quotient” of `mr` and `m0`.
3. Verify that `m0 mp q` equals `mr`.
4. Compare `q` with `r FROM 0 c`.

D2 Enter expressions to obtain the matrices `s1` and `s2` that are Stirling numbers of order 6 (that is, `$ s1` is `6 6`).

```
[ c=. 6?9      s1=_1 FROM 0 c      s2=.0 FROM _1 c ]
```

D3 Test the assertion that `s1` is the inverse of `s2`.

E. SLOPES AS LINEAR FUNCTIONS

As discussed in §1E, a linear function can be represented by a matrix bonded with the matrix product. For example, if `v` is a vector and `ag=. <:/~@i.0#`, then `sum=. ag v` is a *summation* or *aggregation* matrix, in that the linear function (`mp=. +/ . *)&sum` produces sums over prefixes of its argument. Thus:

```

]v=. ^%3 i. 6
0 1 8 27 64 125

```

```

mp=. +/ . *
ag=. <:/~@i.0#
sum=. ag v

```

```

sum
1 1 1 1 1 1
0 1 1 1 1 1
0 0 1 1 1 1
0 0 0 1 1 1
0 0 0 0 1 1
0 0 0 0 0 1

```

```

v mp sum
0 1 9 36 100 225

```

```

+/\v
0 1 9 36 100 225

```

```

mp&sum v
0 1 9 36 100 225

```

```

L1=. mp&sum
L1 v
0 1 9 36 100 225

```

```

sum mp sum
1 2 3 4 5 6
0 1 2 3 4 5
0 0 1 2 3 4
0 0 0 1 2 3
0 0 0 0 1 2
0 0 0 0 0 1

```

```

v mp (sum mp sum)
0 1 10 46 146 371

```

```

+/\ +/\v
0 1 10 46 146 371

```

```

mp&(sum mp sum) v
0 1 10 46 146 371

```

```

L2=. mp&(sum mp sum)
L2 v
0 1 10 46 146 371

```

The results of `L1 v` are rough approximations to the areas under the graph of `*`, that is, to the integrals up to successive points.

Similarly, the inverse matrix `dif=. %.` `sum` can define a linear function that produces differences between successive elements of its argument. For example:

```

dif=. %. sum
dif
1 -1 0 0 0 0
0 -1 -1 0 0 0
0 0 -1 -1 0 0
0 0 0 -1 -1 0
0 0 0 0 -1 -1
0 0 0 0 0 -1

```

```

LD1=. mp&dif
LD1 v
0 1 7 19 37 61

```

```

dif mp dif
1 -2 1 0 0 0
0 -1 -2 1 0 0
0 0 -1 -2 1 0
0 0 0 -1 -2 1
0 0 0 0 -1 -2
0 0 0 0 0 -1

```

```

LD2=. mp&(dif mp dif)
LD2 v
0 1 6 12 18 24

```

These results may be compared with the *l*-slopes of the cube function, noting that the first *k* elements of the *k*th slope are meaningless. The *r*-slopes of a function *f* can be obtained similarly, by ap-

plying $\%r@LD1$ to the results of f applied to arguments differing by r . For example:

```

]x=. r*i.6 [ r=. 0.1
0 0.1 0.2 0.3 0.4 0.5

%r@LD1 ^3 x
0 0.01 0.07 0.19 0.37 0.61

S=. '%x. @ (x.+ -&y. )' : 2
r S (^3) x
0.01 0.07 0.19 0.37 0.61 0.91

```

Except for a displacement of elements due to the meaningless first element in the result of the linear function, its result agrees with the r -slope produced by the slope conjunction s from §B of Chapter 1.

Because the results for the l -slope are so easily extended to the case of a general r -slope, we will discuss only the l -slope provided by the linear function $DIF=. mp&dif$.

Consider the successive applications of DIF to the identity matrix:

```

ID=. (i. =/ i.) 6
DIF=. mp&dif

ID          DIF ID          DIF DIF ID
1 0 0 0 0 0    1 1 0 0 0 0    1 2 1 0 0 0
0 1 0 0 0 0    0 1 1 0 0 0    0 1 2 1 0 0
0 0 1 0 0 0    0 0 1 1 0 0    0 0 1 2 1 0
0 0 0 1 0 0    0 0 0 1 1 0    0 0 0 1 2 1
0 0 0 0 1 0    0 0 0 0 1 1    0 0 0 0 1 2
0 0 0 0 0 1    0 0 0 0 0 1    0 0 0 0 0 1

2 3$ <"2@ (DIF^:0 1 2 3 4 5) ID

```

1 0 0 0 0 0	1 1 0 0 0 0	1 2 1 0 0 0
0 1 0 0 0 0	0 1 1 0 0 0	0 1 2 1 0 0
0 0 1 0 0 0	0 0 1 1 0 0	0 0 1 2 1 0
0 0 0 1 0 0	0 0 0 1 1 0	0 0 0 1 2 1
0 0 0 0 1 0	0 0 0 0 1 1	0 0 0 0 1 2
0 0 0 0 0 1	0 0 0 0 0 1	0 0 0 0 0 1
1 3 3 1 0 0	1 4 6 4 1 0	1 5 10 10 5 1
0 1 3 3 1 0	0 1 4 6 4 1	0 1 5 10 10 5
0 0 1 3 3 1	0 0 1 4 6 4	0 0 1 5 10 10
0 0 0 1 3 3	0 0 0 1 4 6	0 0 0 1 5 10
0 0 0 0 1 3	0 0 0 0 1 4	0 0 0 0 1 5
0 0 0 0 0 1	0 0 0 0 0 1	0 0 0 0 0 1

The foregoing results suggest that the k -th difference is a weighted sum of $k+1$ elements in which the weights are the alternating binomial coefficients of order k . For example:

```
jv=. ^&3 i. 8
0 1 8 27 64 125 216 343
```

```
w=. mp & 1 _2 1
```

```
w 0 1 2{v
```

```
6
```

```
w 1 2 3{v
```

```
12
```

```
w 2 3 4{v
```

```
18
```

```
w 3 4 5{v
```

```
24
```

3 <\ v Box applied to each 3-element window

0 1 8	1 8 27	8 27 64	27 64 125	64 125 216	125 216 343
-------	--------	---------	-----------	------------	-------------

3 w\ v Weighting function applied to each 3-element window
6 12 18 24 30 36

4 (mp & _1 3 _3 1)\ v The third difference of the cube
6 6 6 6 6 function is the constant !3

6 (mp & _1 5 _10 10 _5 1)\ ^&5 i. 11
120 120 120 120 120 120

The binomial coefficients of order n are provided by the expression $(i. n+1) !n$. For example:

```
(i.@>: ! ] ) n=. 5
1 5 10 10 5 1
```

The alternating coefficients could be obtained by multiplying alternate elements by $\_1$. However, they are provided more directly by the extension of the function $!$ to negative arguments, as may be seen in the following function table:

```
ji=. (i.@>:@+: - ] ) 7
_7 _6 _5 _4 _3 _2 _1 0 1 2 3 4 5 6 7
over=. ({.;}.)@":@,
by=. ' ' & @, . @[, .]
```

i by i over i !/ i

	<u>7</u>	<u>6</u>	<u>5</u>	<u>4</u>	<u>3</u>	<u>2</u>	<u>1</u>	0	1	2	3	4	5	6	7
<u>7</u>	1	<u>6</u>	15	<u>20</u>	15	<u>6</u>	1	0	0	0	0	0	0	0	0
<u>6</u>	0	<u>1</u>	<u>5</u>	<u>10</u>	<u>10</u>	<u>5</u>	<u>1</u>	0	0	0	0	0	0	0	0
<u>5</u>	0	0	<u>1</u>	<u>4</u>	<u>6</u>	<u>4</u>	<u>1</u>	0	0	0	0	0	0	0	0
<u>4</u>	0	0	0	<u>1</u>	<u>3</u>	<u>3</u>	<u>1</u>	0	0	0	0	0	0	0	0
<u>3</u>	0	0	0	0	<u>1</u>	<u>2</u>	<u>1</u>	0	0	0	0	0	0	0	0
<u>2</u>	0	0	0	0	0	<u>1</u>	<u>1</u>	0	0	0	0	0	0	0	0
<u>1</u>	0	0	0	0	0	0	<u>1</u>	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	<u>7</u>	<u>6</u>	<u>5</u>	<u>4</u>	<u>3</u>	<u>2</u>	<u>1</u>	0	1	2	3	4	5	6	7
2	<u>28</u>	<u>21</u>	<u>15</u>	<u>10</u>	<u>6</u>	<u>3</u>	<u>1</u>	0	0	1	3	6	10	15	21
3	<u>84</u>	<u>56</u>	<u>35</u>	<u>20</u>	<u>10</u>	<u>4</u>	<u>1</u>	0	0	0	1	4	10	20	35
4	<u>210</u>	<u>126</u>	<u>70</u>	<u>35</u>	<u>15</u>	<u>5</u>	<u>1</u>	0	0	0	0	1	5	15	35
5	<u>462</u>	<u>252</u>	<u>126</u>	<u>56</u>	<u>21</u>	<u>6</u>	<u>1</u>	0	0	0	0	0	1	6	21
6	<u>924</u>	<u>462</u>	<u>210</u>	<u>84</u>	<u>28</u>	<u>7</u>	<u>1</u>	0	0	0	0	0	0	1	7
7	<u>1716</u>	<u>792</u>	<u>330</u>	<u>120</u>	<u>36</u>	<u>8</u>	<u>1</u>	0	0	0	0	0	0	0	1

Except for a change of sign required for the those of odd order, the required alternating binomial coefficients can be seen in the diagonals beginning in row 0 of the negative columns of the foregoing table. The required weights are therefore given by the following function:

```
w=. 1&^ * (i. ! i. - 1)@>:"0
w 0 1 2 3 4
1 0 0 0 0
1 1 0 0 0
1 2 1 0 0
1 3 3 1 0
1 4 6 4 1
```

Differences may therefore be expressed as shown in the following examples:

```
]v=. ^&3 i. 8
0 1 8 27 64 125 216 343
2 mp & (w 1)\ v
1 7 19 37 61 91 127
3 mp & (w 2)\ v
6 12 18 24 30 36
5 mp & (w 4)\ ^&6 i. 10
1560 3360 5880 9120 13080 17760
```

```

rS=. 1 S
^%6 rS rS rS rS i. 6
1560 3360 5880 9120 13080 17760

```

It may also be noted that the diagonals beginning in row 0 of the *non-negative* columns of the table contain the weights appropriate to successive integrations as, for example, in the diagonals beginning with 1 1 1 1 1 and 1 2 3 4 5 and 1 3 6 10 15. This fact can be used to unite the treatment of derivatives and integrals in what Oldham and Spanier call *differintegrals* in their *Fractional Calculus* [6]. Moreover, the fact that the function $!$ is generalized to non-integer arguments will be used (in Chapter 5) to define fractional derivatives and integrals. For example:

```

(i.7)!4
1 4 6 4 1 0 0
8.4":(0.01+i.7)!4
1.0210 4.0333 5.9998 3.9666 0.9793 _0.0020 0.0003

```

Chapter 5: FRACTIONAL CALCULUS

A. INTRODUCTION

The differential and the difference calculus of Chapters 2 and 4 concern derivatives and integrals of integer order. The fractional calculus treated in this chapter unites the derivative and the integral in a single *differintegral*, and extends its domain to non-integer orders.

Section C of Chapter 1 concluded with a brief statement of the utility of the fractional calculus and a few examples of fractional derivatives and integrals. Section E of Chapter 4 concluded with the use of the alternating binomial coefficients produced by the *outof* function ! to compute differences of arbitrary integer order. The extension of the function ! to non-integer arguments was also cited as the basis for an analogous treatment of non-integer differences, and therefore as a basis for approximating non-integer differintegrals.

Our treatment of the fractional calculus will be based on Equation 3.2.1 on page 48 of OS (Oldham and Spanier [6]). Thus:

$f = . \wedge 3$	Function treated
$q = . 2$	Order of differintegral
$N = . 100$	Number of points used in approximation
$a = . 0$	Starting point of integration
$x = . 3$	Argument
$OS = . ' + / (s^{-q}) * (j! j - 1 + q) * f \ x - (s = . N \% \sim x - a) * j = . i . N'$	
" . OS	Execute the Oldham Spanier expression to obtain the
17.82	approximation to the second derivative of f at x
$q = . 1$	
" . OS	Approximation to first derivative (exact value
26.7309	is $3 * x^2$, that is, 27)
$q = . 0$	
" . OS	Zeroth derivative (the function itself)
27	
$q = . \frac{1}{2}$	
" . OS	The first integral (exact value is $\frac{4}{3} * x^4$)
20.657	
$q = . \frac{2}{3}$	
" . OS	The second integral (exact value is $\frac{20}{3} * x^5$)
12.7677	
$q = . 0.5$	
" . OS	
27.9682	Semi-derivative (exact value is 28.1435)

We will use the expression OS to define a *fractional differintegral* conjunction fd such that $q(a, N)$ fd f x produces an N-point approximation to the q-th derivative of the function f at x if $q > 0$, and the $(|q|)$ -th integral from a to x if q is negative:

```

a=. ( ) - 0%{0[] % {1%{0[]
j=. 1.0).
m=. '+/0{(x.%#0]^~0[]*(j0x. !j0x.->:0[])*y.0[]-x.%#0]*j0x.))'
fd=. m : 2

```

For example:

```

2 (0,100) fd (^%3) 3
17.82
2 (0,100) fd (^%3)"0 (0 1 2 3)
0 5.94 11.88 17.82

```

An approximation to a derivative given by a set of N points will be better over shorter intervals. For example:

```

x=. 6
1 (0,100) fd f x
106.924
3*x^2
108
1 ((x-0.01),100) fd f x
107.998

```

Anyone wishing to study the OS formulation and discussion will need to appreciate the relation between the function ! used here, and the gamma function (Γ) used by OS. Although the gamma function was known to be a generalization of the factorial function on integer arguments, it was not defined to agree with it on integers. Instead, Γn is equivalent to $(n-1)!$. Moreover, the dyadic case $m!n$ is here defined as $(n)! * (m)! * (n-m)!$; the three occurrences of the gamma function in Equation 3.2.1 of OS may therefore be written as $j!j-1+q$, as seen in the expression OS used above.

The related *complete beta* function is also used in OS, where it is defined (page 21) by $B(p,q) = (\Gamma p) * (\Gamma q) / (\Gamma p+q)$. This definition may be re-expressed so as to show its relation to the binomial coefficients, by substituting m for p-1 and n for p+q-1. The expression $B(p,q)$ is then equivalent to $(m)! * (n-m)! / (n)!$, or simply $m!n$.

B. TABLE OF SEMI-DIFFERINTEGRALS

The differintegrals of the sum $f+g$ and the difference $f-g$ are easily seen to be the sums and differences of the corresponding differintegrals, and it might be expected that fractional derivatives satisfy further relationships analogous to those shown in §2B for the differential calculus. Such relations are developed by Oldham and Spanier, but most are too complex for treatment here.

We will confine attention to a few of their semi-differintegrals (of orders that are integral multiples of 0.5 and $\frac{1}{2}$). We begin by defining a conjunction FD (similar to fd , but with the parameters a and N fixed at 0 and 100), and using it to define adverbs for approximating semi-derivatives and semi-integrals:

```
FD=. 'x."0 (0 100) fd y. ]' : 2 ("0)
x=. 1 2 3 4 5
1 FD (^&3) x
2.9701 11.8804 26.7309 47.5216 74.2525
3*x^2                      Exact expression
3 12 27 48 75
si=. _1r2 FD                _1r2 is the rational constant _1%2
sd=. 1r2 FD
s3i=. _3r2 FD
^&3 sd x                    Semi-derivative of cube
1.79416 10.1493 27.9682 57.4131 100.296
sdc=. *:0!0[ * (4%*0) ^ [) % !0+:0[ * %:0o.0]
3 sdc x                     Exact expression from OS[6] page 119
1.80541 10.2129 28.1435 57.773 100.925
^&3 si x                    Semi-integral of cube
0.520349 5.88707 24.3343 66.6046 145.442
sic=. *:0!0[ * (4%*0) ^ +&0.50[) % !0>:0+:0[ * %:0o.01:
3 sic x                     Exact function from OS[6] page 119
0.51583 5.83596 24.123 66.0263 144.179
```

Although the conjunctions sd and si and s3i provide only rough approximations, we will use them in the following table to denote exact conjunctions for the semi-differintegrals. This makes it possible to use the expressions in computer experiments, remembering, of course, to wrap any fork in parentheses before applying it.

Function	Semi-derivative	Semi-integral
$f+g$	$f \text{ sd} + g \text{ sd}$	$f \text{ si} + g \text{ si}$
$f-g$	$f \text{ sd} - g \text{ sd}$	$f \text{ si} - g \text{ si}$
$] * g$	$(] * g \text{ sd}) + - : @ (g \text{ si})$	$(] * g \text{ si}) -- : @ (g \text{ s3si})$
$c^0 * g$	$c^0 * g \text{ sd}$	$c^0 * g \text{ si}$
c^0	$c^0 \% \% : @ o.$	$(2 * c)^0 \% \% : @ (] \% 1 p1^0)$
1^0	$\% \% : @ o.$	$+ : @ \% : @ \% @ o. @ \%$
$]$	$+ : @ \% : @ \% @ o. @ \%$	$4 r3^0 * (^{\wedge} 3 r2) \% 1 p1 r2^0$
$*$	$8 r3^0 * (^{\wedge} 3 r2) \% 1 p1 r2^0$	$16 r15^0 * (^{\wedge} 5 r2) \% 1 p1 r2^0$
$\%$	$1 r2 p1 r2^0$	$- : @ (] * 1 p1 r2^0)$
$\% @ > :$	$(\% @ > : - \% : * \_5 \& o. @ \% :) . \% \% : @ o. * > : ^{\wedge} 3 r2^0$	$+ : @ (\_5 \& o. @ \% :) \% \% : @ (> : * 1 p1^0)$
$\% @ \% :$	0^0	$\% : @ (1 p1^0)$
$\% : @ > :$	$1 p1 r2^0 \% \sim \% : @ \% + \_3 \& o. @ \% :$	$1 p1 r2^0 \% \sim \% : + > : * \_3 \& o. @ \% :$
$\% @ \% : @ > :$	$\% @ (> : * \% : * 1 p1 r2^0)$	$+ : @ (\_3 \& o.) @ \% : \% 1 p1 r2^0$
$^{\wedge} \& p$	$\% / @ ! @ ((p-0 \ 1 r2)^0) * ^{\wedge} \& (p-1 r2)$	$\% / @ ! @ ((p+0 \ 1 r2)^0) * ^{\wedge} \& (p+1 r2)$
$^{\wedge} \& n$	$* : @ ! @ (n^0) * ^{\wedge} \& n @ 4 : \% ! @ + : @ (n^0) * \% : @ o.$	$* : @ ! @ (n^0) * ^{\wedge} \& (n+1 r2) @ 4 : \% ! @ > : @ + : @ (n^0) * 1 p1 r2^0$
$^{\wedge} \& (n+1 r2)$	$! @ > : @ + : @ (n^0) * 1 p1 r2^0 * ^{\wedge} \& n @ (1 r4 \& *) \% + : @ * : @ ! @ (n^0)$	$! @ > : @ > : @ + : @ (n^0) * 1 p1 r2^0 * ^{\wedge} \& (> : n) @ (\& 4) \% * : @ ! @ > : @ (n^0)$
$\_3 \& o. @ \% :$	$- : @ \% : @ (1 p1^0 \% > :)$	$1 p1 r2^0 * \% : \& . > :$

Notes:

- f Function
- g Function
- n Integer
- p Constant greater than $\_1$
- c Constant

To experiment with entries in the foregoing table, first enter the definitions of *sd* and *si* and *s31*, and definitions for *f* and *g* (such as *f*= . ^&3 and *g*= . ^&2). The first row would then be treated as:

```
(f+g) sd x=. 1 2 3 4
3.29303 14.3887 35.7565 69.404
(f sd + g sd) x
3.29303 14.3887 35.7565 69.404
(f+g) si x
1.12591 9.31267 33.7741 85.9827
(f si + g si) x
1.12591 9.31267 33.7741 85.9827
```

Entries in the table can be rendered more readable to anyone familiar only with conventional notation by a few assignments such as:

```
twice=. +:
sqrt=. %:
pitimes=. o.
reciprocal=. %
on=. @
```

The table entry for the semi-derivative of the identity function could then be expressed as follows:

```
] sd x
1.12697 1.59378 1.95197 2.25394
twice on sqrt on reciprocal on pitimes on
reciprocal x
1.12838 1.59577 1.95441 2.25676
```

Alternatively, it can be expressed using the *under* conjunction as follows:

```
under=. &.
twice on sqrt on (pitimes under reciprocal) x
1.12838 1.59577 1.95441 2.25676
```


Chapter 6: PROPERTIES OF FUNCTIONS

A. INTRODUCTION

In this chapter we will analyze relations among the functions developed in Chapter 2, and will express them all as members of a single family. The general approach will be to first attempt to discover interesting relations by experimentation, and then to construct proofs. In this section we will use the growth and decay functions to illustrate the process; we will then devote separate sections to experimentation and to proof. We will use the definition $D=.$ ("0) (D.1) .

The reader is urged to try to develop her own experiments before reading Section B, and her own proofs before reading Section C.

In Sections E and F of Chapter 2, the functions *ec* and *eca* were developed to approximate growth and decay functions. Thus:

```
ec=. %0!  
eca=. _1&^ * ec  
ec i.7  
1 1 0.5 0.166667 0.0416667 0.00833333 0.00138889  
eca i.7  
1 _1 0.5 _0.166667 0.0416667 _0.00833333 0.00138889
```

We will now use the approximate functions to experiment with growth and decay:

```
GR=. (ec i.20)&p=. #.&|.~"1 0  
DE=. (eca i.20)&p
```

It might be suspected that the decay function would be the reciprocal of the growth function, in other words that their product is *one*. We will test this conjecture in two ways, first by computing the product directly, and then by computing the coefficients of the corresponding product polynomial. Thus:

```
GR x=. 0 1 2 3 4  
1 2.71828 7.38906 20.0855 54.5981  
DE x  
1 0.367879 0.135335 0.0497871 0.0183153  
(GR x) * (DE x)  
1 1 1 1 0.999979  
(GR * DE) x  
1 1 1 1 0.999979  
PP=. +//.0(* /)  
1 2 1 PP 1 3 3 1  
1 5 10 10 5 1
```

```

6{. (ec i.20) PP (eca i.20)
1 0 0 1.35525e_20 3.38813e_21 8.47033e_22
6{. (ec PP eca) i.20
1 0 0 1.35525e_20 3.38813e_21 8.47033e_22
((ec PP eca) i.20) p x
1 1 1 1 0.999979

```

Since the growth and decay functions were defined only in terms of their derivatives, any proof of the foregoing conjecture must be based on these defining properties. We begin by determining the derivative of the product as follows:

```

(DE*GR) D
(DE*GR D )+(DE D *GR)      θ 4, Table B1, Chapter 2
(DE*GR)+(DE D *GR)         Definition of GR
(DE*GR)+(-θDE*GR)          Definition of DE
(DE*GR)-(DE*GR)
0:"0

```

Consequently, the derivative of $DE*GR$ is zero; $DE*GR$ is therefore a constant, whose value may be determined by evaluating the function at any point. At the argument 0, all terms of the defining polynomials are zero except the first. Hence the constant value of $DE*GR$ is *one*, and it is defined by the function $1:"0$. Thus:

```

(DE*GR) x      1:"0 x
1 1 1 1 0.999979      1 1 1 1 1

```

A second experiment is suggested by the demonstration (in §2I) that the derivative of the function $f = \wedge \theta(x \&*)$ is x times f ; the case $x = \_1$ should give the decay function:

```

r=. _1
DE x=. 0 1 2 3 4
1 0.367879 0.135335 0.0497871 0.0183153
^θ(x&*) x
1 0.367879 0.135335 0.0497871 0.0183156
^ AM x x
1 0.367879 0.135335 0.0497871 0.0183156

```

The final expression uses the scaling conjunction of §2I. We may now conclude that the function $\wedge AM x$ describes growth at any rate, and that negative values of x subsume the case of decay.

In the foregoing discussion we have used simple observations (such as the probable reciprocity of growth and decay) to motivate experi-

ments that led to the statement and proof of significant identities. To any reader already familiar with the exponential function these matters may seem so obvious as to require neither suggestion nor proof, and he may therefore miss the fact that all is based only on the bare definitions given in Sections 2E and 2F.

Similar remarks apply to the hyperbolic and circular functions treated in Sections 2G,H. The points might be better made by using featureless names such as $\mathbf{f1}$, $\mathbf{f2}$, and $\mathbf{f3}$ for the functions. However, it seems better to adopt commonly used names at the outset.

A1 Test the proof of this section by entering each expression with an argument.

A2 Make and display the table $\mathbf{T}$ whose (counter) diagonal sums form the product of the coefficients $\mathbf{ec\ i.7}$ and $\mathbf{eca\ i.7}$.

[$\mathbf{T=. (ec\ */\ eca)\ i.7}$]

A3 Denoting the elements of the table $\mathbf{t=. 2\ 2\{.T}$ by $\mathbf{t00}$, $\mathbf{t01}$, $\mathbf{t10}$, and $\mathbf{t11}$, write explicit expressions for them. Then verify that $\mathbf{t00}$ and $\mathbf{t01+t10}$ agree with the first two elements of the product polynomial given in the text.

[$\mathbf{t00\ is\ 1*1\ \quad t01+t10\ is\ (1*_1)+(1*1)}$]

A4 Use the scheme of A3 on larger subtables of $\mathbf{T}$ to check further elements of the polynomial product.

A5 Repeat the exercises of this section for other relations between functions that might be known to you.

[Consider the functions $\mathbf{f=. ^\star ^}$ and $\mathbf{g=. ^\theta+}$: beginning by applying them to various arguments, as in $\mathbf{f"0\ i.5}$ and $\mathbf{g"0\ i.5}$]

B. EXPERIMENTATION

Hyperbolics. One hyperbolic may be plotted against the other by using the conjunction $\mathbf{vs}$ of §1E. For example:

$\mathbf{sinh=. 5\&0.}$

$\mathbf{cosh=. 6\&0.}$

$\mathbf{N=. '(\%>./"1)\@((((>./-])\@x. ,: \{-<./)\@y.)\@))' : 22}$

$\mathbf{r=. [\#.:@(<.\@(\%2+))\@(<:\@(\%>./"1)\@))}$

$\mathbf{vs=. 'i.\@[e. [r\ x. N\ y.' : 22}$

' *' (~ 18 30 sinh VS cosh a = . 0.2*10--i.21

The resulting plot suggests an hyperbola that satisfies the equation $1 = (\sqrt{x}) - (\sqrt{y})$. Thus:

Finally, each of the hyperbolics is the derivative of the other, and their second derivatives equal the original functions:

Circulars. The circular functions may be plotted similarly:

cosh = cosh **EVEN**

sin = sin **ODD**

cosh = ^ **EVEN**

cos = cos **EVEN**

B1 Repeat Exercises A2-A5 with modifications appropriate to the circular and hyperbolic functions.

C. PROOFS

We will now use the definitions of the hyperbolic and circular functions to establish the two main conjectures of Section B:

cosh -&*: **sinh** is 1

cos +&*: **sin** is 1

The proof parallels that used for **DE*GR** in § A:

(**cosh** -&*: **sinh**) **D**

(*: @**cosh** - *: @**sinh**) **D**

Def of &

(*: **D** @**cosh*****cosh** **D**) - (*: **D** @**sinh*****sinh** **D**) θ 5

(*: **D** @**cosh** * **sinh**) - (*: **D** @**sinh** * **cosh**) See §2G

(2&*@**cosh** * **sinh**) - (2&*@**sinh** * **cosh**) θ 6 (n=. 2)

(2&[* **cosh** * **sinh**) - (2&[* **sinh** * **cosh**)

2&[* (**cosh** * **sinh**) - (**sinh** * **cosh**)

0:"0

The circular case differs only in the values for the derivatives:

cos **D** is -@**sin**

sin **D** is **cos**

C1 Write and test a proof of the fact that the sum of the squares of the functions 1&o. and 2&o. is 1.

D. THE EXPONENTIAL FAMILY

We have now shown how the growth, decay, and hyperbolic functions can be expressed in terms of the single exponential function ^ :

^ **AM** **r** Growth at rate **r**

^ **EVEN** Hyperbolic cosine

^ **ODD** Hyperbolic sine

Complex numbers can be used to add the circular functions to the exponential family as follows:

^@j. **EVEN** Cosine

^@j. ODD Sine multiplied by 0j1

For example:

```

^@j. EVEN x=. 0 1 2 3 4
1 0.540302 _0.416147 _0.989992 _0.653644
cos x
1 0.540302 _0.416147 _0.989992 _0.653644
^@j. ODD x
0 0j0.841471 0j0.909297 0j0.14112 0j_0.756802
j. sin x
0 0j0.841471 0j0.909297 0j0.14112 0j_0.756802
j.^:_1 ^@j. ODD x
0 0.841471 0.909297 0.14112 _0.756802
^ ODD &. j. x
0 0.841471 0.909297 0.14112 _0.756802

```

D1 Write and test tautologies involving *cosh* and *sinh*.

[t=. cosh = sinh@j. and u=. sinh = cosh@j.]

D2 Repeat D1 for *cos* and *sin*.

E. LOGARITHM AND POWER

The inverse of the exponential is called the *logarithm*, or *natural logarithm*. It is denoted by $\wedge.$; some of its properties are shown below:

```

D=. ("0) (D.1)                      Scalar derivative adverb
I=. ^:_1                              Inverse adverb
^ I x=. 1 2 3 4 5
0 0.693147 1.09861 1.38629 1.60944
^ ^ I x
1 2 3 4 5
^ . x                                  Natural log
0 0.693147 1.09861 1.38629 1.60944
^ . D x
1 0.5 0.333333 0.25 0.2
% ^ . D x
1 2 3 4 5
^ . x ^ b=. 3
0 2.07944 3.29584 4.15888 4.82831

```

```

      b * ^ . x
0 2.07944 3.29584 4.15888 4.82831

```

The dyadic case of the logarithm $\wedge$ is defined in terms of the monadic as illustrated below:

```

      (^ . x) % (^ . b)
0 0.63093 1 1.26186 1.46497

      b ^ . x
0 0.63093 1 1.26186 1.46497

      b %&^ . ~ x
0 0.63093 1 1.26186 1.46497

```

The dyadic case of $\wedge$ is the power function; it has, like other familiar dyads (including $+$ $-$ $*$ $\%$) been used without definition. We now define it in terms of the dyadic logarithm as illustrated below:

```

      b&^ . I x      (Where I=. ^:_1)
3 9 27 81 243

      b ^ x
3 9 27 81 243

```

This definition extends the domain of the power function beyond the non-negative integer right arguments embraced in the definition of power as the product over repetitions of the left argument, as illustrated below:

```

      m=. 1.5
      n=. 4
      n # m
1.5 1.5 1.5 1.5

      */ n # m
5.0625

      m^n
5.0625

```

Moreover, the extended definition retains the familiar properties of the simple definition. For example:

```

      5 ^ 4+3          (5^4) * (5^3)
78125                78125

```

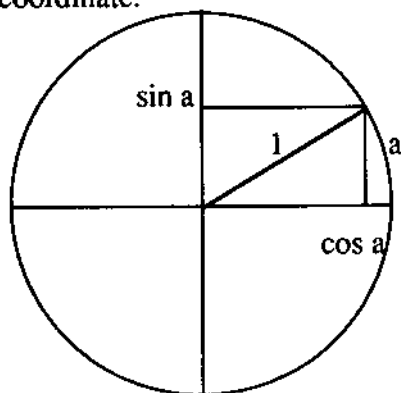
E1 Comment on the question of whether the equivalence of $*/n\#m$ and m^n holds for the case $n=.0$.

F. TRIGONOMETRIC FUNCTIONS

Just as a five-sided (five-angled) figure may be characterized either as *pentagonal* or *pentangular*, so may a three-sided figure be characterized as *trigonal* or *triangular*. The first of these words suggests the etymology of *trigonometry*, the measurement of three-sided figures. This section concerns the equivalence of the functions *sin* and *cos* (that have been defined only by differential equations) and the corresponding trigonometric functions *sine* and *cosine*.

The sine and cosine are also called *circular* functions, because they can be defined in terms of the coordinates of a point on a *unit* circle (with radius 1 and centre at the origin) as functions of the length of arc to the point, measured counter-clockwise from the reference point with coordinates 1 0. As illustrated in Figure F1, the cosine of a is the horizontal (or x) coordinate of the point whose arc is a , and the sine of a is the vertical coordinate.

Figure F1



The length of arc is also called the *angle*, and the ratio of the circumference of a circle to its diameter is called π , given by $\pi \approx 3.14159$, or by the constant 2π . The circular functions therefore have the period 2π , that is *two pi*. Moreover, the coordinates of the end points of arcs of lengths π and 0.5π are $-1, 0$ and $0, 1$; the *supplementary* angle $\pi - a$ and the *complementary* angle $0.5\pi - a$ are found by moving *clockwise* from these points.

Taken together with these remarks, the properties of the circle make evident a number of useful properties of the sine and cosine. We will illustrate some of them below by tautologies, each of which can be tested by enclosing it in parentheses and applying it to an argument, as illustrated for the first of them:

S=. 1&o.

C=. 2&o.

x=. 1 2 3 4 5

(1: = S +&*: C)

Theorem of Pythagoras

1 1 1 1 1

S@- = -@S

The sine is odd

S ODD = S

C@- = C

The cosine is even

S @ (2p1&+) = S

The period of the sine is twice pi

C @ (2p1&+) = C

The period of the cosine is twice pi

S @ (1p1&-) = S

Supplementary angles

C @ (1p1&-) = -@C

" "

S @ (0.5p1&-) = C

Complementary angles

C @ (0.5p1&-) = S

" "

Sum Formulas. A function applied to a sum of arguments may be expressed equivalently in terms of the function applied to the individual arguments; the resulting relation is called a *sum formula*:

a=. 2 3 5 7

b=. 4 3 2 1

+: a+b

12 12 14 16

(+:a)+(+:b)

12 12 14 16

*: a*b

36 36 49 64

(*:a)+(+:a*b)+(*:b)

36 36 49 64

^ a^b

403.429 403.429 1096.63 2980.96

(^a)*(^b)

403.429 403.429 1096.63 2980.96

Sum formulas may also be expressed as tautologies:

$$+:@+ = +:@[+ +:@]$$

$$a(+:@+ = +:@[+ +:@]) \ b$$

1 1 1 1

$$*:@+ = *:@[+ +:@* + *:@]$$

$$^@+ = ^@[* ^@]$$

The following sum formulas for the sine and cosine are well known in trigonometry:

$$s@+ = (s@[* c@]) + (c@[* s@])$$

$$s@- = (s@[* c@]) - (c@[* s@])$$

$$c@+ = *sC - *sS$$

$$c@- = *sC + *sS$$

Since $a \ s@+ \ a$ is equivalent to (the monadic) $s@+:$, we may obtain the following identities for the double angle:

$$s@+: = +:@(S * C)$$

$$c@+: = *:@C - *:@S$$

The theorem of Pythagoras can be used to obtain two further forms of the identity for $c@+:$:

$$c@+: = -.@+:@*:@S$$

$$c@+: = <:@+:@*:@C$$

An identity for the sine of the half angle may be obtained as follows:

$$c@+:@-: = -.@+:@*:@s@-:$$

$$C = -.@+:@*:@s@-:$$

$$+:@*:@s@-: = -.@C$$

$$s@-: = \&| (+:@*:@ \ 1.1) @-.@C$$

$$s@-: = \&| \>:@-:@-.@C$$

The last two tautologies above compare *magnitudes* ($=\&|$) because the square root yields only the positive of the two possible roots. Similarly for the cosine:

$$c@+:@-: = <:@+:@*:@c@-:$$

$$C = <:@+:@*:@c@-:$$

$$c@-: = \&| \>:@-:@>:@C$$

Tautologies may be re-expressed in terms of arguments i and x as illustrated below for $s@+$ and $c@+$:

$$(S \ i+x) = ((S \ i)*(C \ x)) + ((C \ i)*(S \ x))$$

$$(C \ i+x) = ((C \ i)*(C \ x)) - ((S \ i)*(S \ x))$$

Derivatives. We may express the secant slope of the sine function at the points x and $x+i$ as follows:

$$\frac{(S(x+i) - S(x))}{i}$$

Using the sum formula for the sine we obtain the following equivalent expressions:

$$\frac{((S(x) \cos i) + (\cos x) S(i) - S(x))}{i}$$

$$\frac{((S(x) \cos i) + S(x) (-\sin i))}{i}$$

$$\frac{((S(x) \cos i) - S(x) \sin i) - S(x)}{i}$$

$$\frac{(\cos x) S(i) - S(x) \sin i}{i}$$

To obtain the derivative of s from this secant slope, it will be necessary to obtain limiting values of the ratios $\frac{S(i)}{i}$ and $\frac{(-\cos i)}{i}$.

In the unit circle of Figure F2, the magnitude of the area of the sector with arc length (angle in radians) i lies between the areas of the triangles OSC and OST. Moreover, the lengths of the relevant sides are as shown below:

OC	CS	OS	ST
$\cos i$	$\sin i$	1	$(\tan i)$

ST is the tangent to the circle, and its length is called the *tangent* of i . Its value $(\tan i)$ follows from the ratios in the similar triangles.

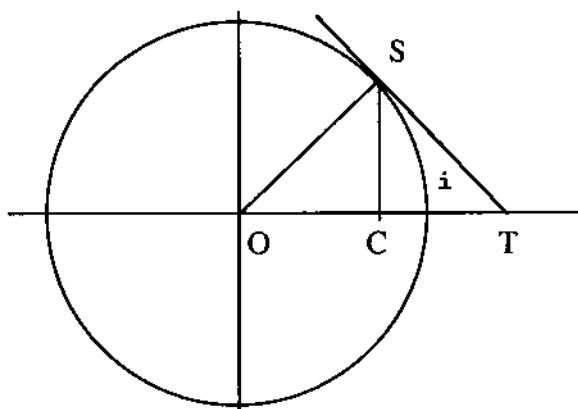


Figure F2

The values of the cited areas are therefore $-\theta(s^*c)$ and $-\theta i$ and $-\theta(s^*c) i$. Multiplying by 2 and dividing by $s i$ gives the relative sizes $c i$ and $i s i$ and $c i$. Hence, the ratio $i s i$ lies between $c i$ and $c i$, both of which are 1 if $i = 0$. Finally, the desired limiting ratio $(s i) i$ is the reciprocal, also 1.

The limiting value of $(-c i) i$ can be obtained from the identity $+: \theta^*: \theta s \theta^- = -\theta c$, for:

$$\begin{aligned} & (-c i) i \\ & (+: \theta^*: s i^2) i \\ & (\theta^*: s i^2) i (i^2) \\ & ((s i^2) i (i^2)) * (s i^2) \end{aligned}$$

The limit of the first factor has been shown to be 1, and the limit of $s i^2$ is 0; hence the limit of $(-c i) i$ is their product, that is, 0.

Substituting these limiting values in the expression for the secant slope $((c x) * ((s i) i)) - (s x) * ((-c i) i)$ we obtain the expression for the derivative of the sine, namely:

$$\begin{aligned} & ((c x) * (1)) - (s x) * (0) \\ & c x \end{aligned}$$

Similar analysis shows that the derivative of c is $-\theta s$, and we see that the relations between s and c and their derivatives are the same as those between $\sin$ and $\cos$ and their derivatives. Moreover, the values of s and $\sin$ and of c and $\cos$ agree at the argument 0.

F1 Define $f = . \sin \theta (+/) = \text{perm} \theta : sc$ and $sc = . 1 \ 2 \& \theta . "0$ and $\text{perm} = . +/ . *$ and $\sin = . 1 \& \theta .$ and $\cos = . 2 \& \theta .$; then evaluate $f \ a, b$ for various scalar values of a and b and comment on the results.

[f is a tautology recognizable as

$$(\sin(a+b)) = ((\sin a) * (\cos b)) + ((\cos a) * (\sin b))]$$

F2 Define other tautologies known from trigonometry in the form used in F1. [Consider the use of $\det = . -/ . *$]

G. DOT AND CROSS PRODUCTS

As illustrated in §1C, the vector derivative of the function $*/\backslash$ yields a matrix result; the vectors in this matrix lie in a plane, and the vector

perpendicular or *normal* to this plane is an important derivative called the *curl* of the vector function. We will now present a number of results needed in its definition, including the *dot* or *scalar* product and the *cross* or *vector* product.

The angle between two rays from the origin is defined as the length of arc between their intersections with a circle of unit radius centred at the origin. The angle between two vectors is defined analogously. For example, the angle between the vectors $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is $\frac{1}{4}\pi$ (that is, one-fourth of π) radians, or 45 degrees. If the angle between two vectors is $\frac{1}{2}\pi$ radians (90 degrees), they are said to be *perpendicular* or *normal*.

Similar notions apply in three dimensions, and a vector $\mathbf{r}$ that is normal to each of two vectors $\mathbf{p}$ and $\mathbf{q}$ is said to be normal to the plane defined by them, in the sense that it is normal to every vector of the form $(\mathbf{a} \cdot \mathbf{p}) + (\mathbf{b} \cdot \mathbf{q})$, where $\mathbf{a}$ and $\mathbf{b}$ are scalars.

The remainder of this section defines the dot and cross products, and illustrates their properties. Proofs of these properties may be found in high-school level texts as, for example, in Sections 6.7, 6.8, and 6.12 of Coleman et al [12]. Again we will leave interpretations to the reader, and will defer comment on them to exercises.

The dot product may be defined by $\mathbf{a} \cdot \mathbf{b}$ or, somewhat more generally, by $\mathbf{a} \cdot \mathbf{b}$. Thus:

$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$
$\mathbf{a} \cdot \mathbf{b}$	
16	
$\mathbf{a} \cdot \mathbf{b}$	
16	
$\mathbf{a} \cdot \mathbf{a}$	
14	
$\mathbf{L} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$	
$\mathbf{a}, \mathbf{b}$	
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	
$\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$	
$\mathbf{L} \mathbf{a}, \mathbf{b}$	
3.74166 5.38516	
$\mathbf{a} \cdot \mathbf{L} \mathbf{a}, \mathbf{b}$	
20.1494	
	The product of the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$ with the product of their lengths
	Squared length of $\mathbf{a}$
	Length function
	Product of lengths

```

(a dot b) % */L a,:b
0.794067
cos=. dot % */@ (L@,:) Re-definition of cos (not 2so.)
a cos b
0.794067
0 0 1 cos 0 1 0 Perpendicular or normal vectors
0
0 0 1 cos 0 1 1
0.707107
2 o. 1r4p1 Cosine of 45 degrees
0.707107

```

The following expressions lead to a definition of the cross product and to a definition of the sine of the angle between two vectors:

```

rot=. |."1 Rotation of rank 1
1 _1 rot a
2 3 1
3 1 2
(1 _1 rot a) * (_1 1 rot b)
4 12 3
9 2 8
]c=. -/(1 _1 rot a) * (_1 1 rot b) The cross product
_5 10 _5
a dot c The vectors are each normal to
0 their cross product
b dot c
0
cross=. -/@(1 _1&rot@[ * _1 1&rot@])
a cross b
_5 10 _5
(a,:b) dot a cross b
0 0
b cross a The cross product is not
5 _10 5 commutative
L a cross b The product of the sine of the angle between
12.2474 the vectors with the product of their lengths
(L a cross b) % */ L a,:b The sine of the angle
0.607831
sin=. L@cross % */@ (L@,:) The sine function
a sin b a +/@*:@(sin , cos) b
0.607831 1

```

The following expressions suggest interpretations of the dot and cross products that will be pursued in exercises:

$$\begin{array}{rcl}
 c = . \quad 4 \quad 1 \quad 2 & m = . \quad c, a, : b & - / . * m \\
 c \text{ dot } a \text{ cross } b & m & \_20 \\
 \_20 & 4 \quad 1 \quad 2 & \\
 & 1 \quad 2 \quad 3 & \\
 & 4 \quad 3 \quad 2 &
 \end{array}$$

G1 Experiment with the dot and cross products, beginning with vectors in 2-space (that is with two elements) for which the results are obvious. Continue with other vectors in 2-space and in 3-space. Sketch the rays defined by the vectors, showing their intersection with the unit circle (or sphere).

H. SYMMETRIES

The adverb $\dots$ - introduced in §1 C produces the symmetric part of its argument; that is, $f \dots$ - is defined by $\neg : \theta (f + f \theta \neg)$. It was used in §D to express the hyperbolic cosine as the symmetric part of the exponential.

More generally, $f \dots g$ is defined by $\neg : \theta (f + f \theta g)$ for any monad g , and we would express the hyperbolic cosine more fully as the symmetric part of the exponential *with respect to negation*. Similarly, $f \dots : g$ is defined by $\neg : \theta (f - f \theta g)$, the skew-symmetric part of f with respect to g .

Symmetries with respect to transpositions of axes are of particular interest. For example:

$$\begin{array}{rcl}
 f = . \quad 9 : \quad - \quad * : & m = . \quad i. \quad 3 \quad 3 & \\
 m & f \quad m & \\
 0 \quad 1 \quad 2 & 9 \quad 8 \quad 5 & \\
 3 \quad 4 \quad 5 & 0 \quad 7 \quad 16 & \\
 6 \quad 7 \quad 8 & 27 \quad 40 \quad 55 & \\
 f \dots | : m & f \dots : | : m & \\
 9 \quad 4 \quad 11 & 0 \quad 4 \quad 16 & \\
 4 \quad 7 \quad 28 & 4 \quad 0 \quad 12 & \\
 11 \quad 28 \quad 55 & 16 \quad 12 \quad 0 & \\
 (f \neg : f \dots | : + f \dots : | :) m & & \text{A tautology} \\
 1 & & \\
 n = . \quad i. \quad 2 \quad 2 \quad 2 & &
 \end{array}$$

	n		f n		f .. : n
0	1		9 8		9 0.5
2	3		5 0		5 13.5
4	5		7 16		0.5 16
6	7		27 40		13.5 40

t=. 0 2 1&|:

	f .. t n		(f .. t + f .. t) n
	9 6.5		9 8
	6.5 0		5 0
	7 21.5		7 16
	21.5 40		27 40

A function **f** is said to be skew-symmetric with respect to a transposition **t** if **f .. t** is identical to **-@f**; it is said to be *completely* skew (of rank **r**) if **f"r .. t** is reversed in sign by *any* transposition **t**.

As illustrated in the function **t=. 0 2 1&|:**, an arbitrary permutation of the axes of an array **a** is provided by the expression **p&|:a**, where **p** is a permutation vector of order **r=. #@a** (the rank of **a**); that is, **p** is a re-ordering of the elements of **i. r**.

Any permutation can be effected as a sequence of interchanges of *pairs*. For example, the permutation **p=. 3 0 2 1** can be effected by interchanging the first two elements, followed by a further exchange of the first and last:

```
p=. 3 0 2 1 [x=. 'ABCD'
p C. x
DACB
0 3;0 1          (0 3;0 1) C. x
DACB
0 3 0 1          (0 1;1 2;2 3;1 2) C. x
DACB
```

As illustrated above, a given permutation can be effected by different sequences of pair-exchanges; however, the *number* **n** of exchanges will remain either even or odd, and we define the *parity* of a vector as **_1^n** if it is a permutation, and 0 if it is not. We now define a function for parity, and use it to define a function such that **csk n** produces a completely skew array of shape **n#n**:

```
par=. _1^@(+/)@(>:@(#>)@C.) :: 0:"1
```

```
CSK=. par@indices=. (I #: i.)@(#~)
```

```
<"2 indices 3
```

0 0 0	0 1 0	0 2 0
0 0 1	0 1 1	0 2 1
0 0 2	0 1 2	0 2 2
1 0 0	1 1 0	1 2 0
1 0 1	1 1 1	1 2 1
1 0 2	1 1 2	1 2 2
2 0 0	2 1 0	2 2 0
2 0 1	2 1 1	2 2 1
2 0 2	2 1 2	2 2 2

```
CSK 3
```

```
0 0 0
0 0 1
0 _1 0
```

```
0 0 _1
0 0 0
1 0 0
```

```
0 1 0
_1 0 0
0 0 0
```

```
<"2 CSK 4
```

0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
0 0 0 0	0 0 0 0	0 0 0 _1	0 0 1 0
0 0 0 0	0 0 0 1	0 0 0 0	0 _1 0 0
0 0 0 0	0 0 _1 0	0 1 0 0	0 0 0 0
0 0 0 0	0 0 0 0	0 0 0 1	0 0 _1 0
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
0 0 0 _1	0 0 0 0	0 0 0 0	1 0 0 0
0 0 1 0	0 0 0 0	_1 0 0 0	0 0 0 0
0 0 0 0	0 0 0 _1	0 0 0 0	0 1 0 0
0 0 0 1	0 0 0 0	0 0 0 0	_1 0 0 0
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
0 _1 0 0	1 0 0 0	0 0 0 0	0 0 0 0
0 0 0 0	0 0 1 0	0 _1 0 0	0 0 0 0
0 0 _1 0	0 0 0 0	1 0 0 0	0 0 0 0
0 1 0 0	_1 0 0 0	0 0 0 0	0 0 0 0
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0

The skew array can be used to provide an interesting alternative definition of the determinant. The determinant of a matrix is a weighted sum of the products of elements from different rows. The multiplication table `*/m` contains all such products, and the array `CSK@# m` provides the weights:

```

]m=. *: i. 3 3
0 1 4
9 16 25
36 49 64
*/m
0 0 0
0 0 0
0 0 0
CSK@# m
0 0 0
0 0 1
0 _1 0
324 441 576
576 784 1024
900 1225 1600
0 0 _1
0 0 0
1 0 0
1296 1764 2304
2304 3136 4096
3600 4900 6400
0 1 0
_1 0 0
_0 0 0
+/@,@(CSK@# * */) m -/ . * m
_216 _216

```

H1 Experiment with the expressions of this section.

I. NORMALS

We now develop a function `norm` that is a generalization of the cross product; it applies to arrays other than vectors, and produces a result that is normal to its argument. Moreover, when applied to skew arrays of odd order (having an odd number of items) it is self-inverse. Thus:

```

]skm=. *: .: |: i. 3 3
0 _4 _16
4 _0 _12
16 12 _0
A skew matrix

]v=. -: +/ +/ skm * CSK #skm
_12 16 _4
v +/ . * skm
0 0 0
Test of orthogonality

+/ v * CSK #v
0 _4 _16
4 _0 _12
16 12 _0
Inverse transformation

norm=. +/^: (]^ (#@$) ^ (* CSK@#)) * !@ (#-#@$)

```

```

]m=. (a=. 1 2 3) */ (b=. 4 3 2)
4 3 2
8 6 4
12 9 6

```

n=. norm ^:

An adverb for powers of norm

```

0 n m
4 3 2
8 6 4
12 9 6
1 n m
_5 10 _5
a cross b
_5 10 _5

```

Skew part of m

```

2 n m
0 _2.5 _5
2.5 0 _2.5
5 2.5 0
3 n m
_5 10 _5
1 n a
0 1.5 _1
_1.5 0 0.5
1 _0.5 0
2 n a

```

Self-inverse for odd dimension

```

1 2 3
dot=. +/ . *
a dot 1 n a*/b
0
b dot 1 n a*/b
0

```

```

x=. 1 2
1 n x
_2 1
x mp 1 n x
0

```

For even orders 2 n is inverse
only up to sign change

```

2 n x
_1 _2
4 n x
1 2
2 n y=. 1 2 3 4
_1 _2 _3 _4
4 n y
1 2 3 4

```

```

2 n 1 2 3 4 5
1 2 3 4 5

```

```

cr=. norm@ (*)      Alternative definition of cross product
a cr b
_5 10 _5
a cross b
_5 10 _5

```

I1 Experiment with the expressions of this section.

I2 Using the display of `csk 3` shown in Section H, and using `a0`, `a1`, and `a2` to denote the elements of a vector `a` in 3-space, show in detail that `norm@ (*)` is indeed an alternative definition of the cross product.

I3 Show in detail that `+/@, @ (csk@# * *//)` is an alternative definition of the determinant.

Chapter 7: INTERPRETATIONS AND APPLICATIONS

A. INTRODUCTION

As remarked in §1D, various interpretations of a particular function definition are possible (as in `vol=. */"1` and `cost=. */"1`), and any one of them may be either helpful or confusing, depending upon the background of the reader. A helpful interpretation may also be misleading, either by suggesting too little or too much. We will illustrate this point by three examples.

Example 1. The sentences:

```
s=. '%&x. @ (] -&y. -&x.)' : 2
f=. ^
h=. 1e_8
sf=. h s f
sf x=. 1
```

2.71828

define and use the function `sf`. Moreover, `sf` can be helpfully interpreted as the secant slope of the exponential with spacing `h`, and (because `h` is small) as an approximation to the tangent slope of the exponential.

However, for the case of the discontinuous *integer part* function `<`, this interpretation would be misleading because its “tangent slope” at the point 1 is infinite. Thus:

```
h s <. x
1e8
```

Example 2. If the spacing `h` is complex, the function `h s ^` has the behaviour expected of a secant slope:

```
^ y=. 2j3
_7.31511j1.04274
h=. 1e_6j1e_8
h s ^ y
_7.31511j1.04274
(r=. 1e_6j0) s ^ y
_7.31511j1.04274
(i=. 0j1e_8) s ^ y
_7.31511j1.04274
```

Again the interpretation of the function `h s f` as an approximation to the tangent slope is valid.

However, the (continuous) *conjugate* function $\bar{z}$ shows unusual behaviour:

```

      h s + y
0.9998j_0.019998
      r s + y
1
      i s + y
_1

```

The problem arises because the conjugate is not an *analytic* function. A clear and simple discussion of this matter may be found in Churchill [13].

Example 3.

Section 2D interprets the integral of a function f as a function that gives the area under the graph of f from a point a (that is, the point $a, f(a)$ on the graph of f) to a second point b . This interpretation is helpful for real-valued functions, but how should we visualize the area under a function that gives a complex result?

It is, of course, possible to interpret the integral as a complex result whose real and imaginary parts are the areas under the real and imaginary parts of f , respectively. However, the beginning and end points may themselves be complex, and although there is a clearly defined “path” through real numbers between a pair of real numbers a and b , there are an infinity of different paths through complex numbers from complex a to complex b .

This observation leads to the more difficult, but highly useful, notion of integration along a prescribed path (called a *line* or *contour* integral), a notion not hinted at by the interpretation of integration as the area under a curve.

B. APPLICATIONS AND WORD PROBLEMS

What we have treated as *interpretations* of functions may also be viewed as *applications* of math, or as *word problems* in math. For example, the function:

```

f=. 0.1&path=. (cos,sin)@*"0[. (cos=.2&o.) [. (sin=.1&o.)

```

defined in §1D was *interpreted* as the “Position of a car ... moving on a circular path at an angular velocity of 0.1 radians per second”.

Conversely, the expression in quotes could be considered as an *application* of the circular functions, and could be posed as a *word problem* requiring as its solution a definition of the function f .

Similarly, the phrase $f \cdot D$ was interpreted as the velocity of the car whose position was prescribed by f . Because the phrase involved a derivative, the corresponding word problem would be considered as an application of the calculus.

Just as a reader's background will determine whether a given interpretation is helpful or harmful in grasping new concepts in the calculus, so will it determine the utility of word problems. We will limit our treatment of interpretations and applications to a few examples, and encourage the reader to choose further applications from any field of interest, or from other calculus texts.

C. EXTREMA AND INFLECTION POINTS

If $f = .(c = . 0 1 2.5 \_2 0.25) \&pic = . \#.\sim\&|. "1 0$, then pic is a polynomial in terms of coefficients, and f is a specific polynomial whose (tightly) formatted results:

```
(fmt = . 5.16":) f x = . 0.1*>:i.6 10
0.1 0.3 0.5 0.7 0.9 1.1 1.3 1.5 1.6 1.8
1.8 1.9 1.8 1.8 1.6 1.4 1.2 0.9 0.5 0.0
0.5 1.1 1.8 2.6 3.4 4.2 5.2 6.1 7.2 8.2
9.4 10.5 11.7 12.9 14.1 15.3 16.5 17.7 18.9 20.0
21.1 22.1 23.0 23.9 24.6 25.2 25.7 26.1 26.3 26.2
26.0 25.6 25.0 24.1 22.9 21.4 19.6 17.4 14.9 12.0
```

suggest that it has a (local) maximum (of 1.9) near 1.2 and a minimum near 4.9. Moreover, a graph of the function over the interval from 0 to 4 shows their location more precisely.

A graph of the derivative $f \cdot D = . ("0) (D.1)$ over the same interval illustrates the obvious fact that the derivative is zero at an extremum (minimum or maximum):

```
fmt f (D = . ("0) (D.1)) x
1.4 1.8 2.0 2.1 2.1 2.1 1.9 1.7 1.4 1.0
0.6 0.1 0.4 1.0 1.6 2.3 2.9 3.6 4.3 5.0
5.7 6.4 7.1 7.7 8.4 9.0 9.6 10.1 10.6 11.0
11.4 11.7 11.9 12.1 12.1 12.1 12.0 11.8 11.4 11.0
10.4 9.8 8.9 8.0 6.9 5.6 4.2 2.6 0.9 1.0
3.1 5.4 7.8 10.5 13.4 16.5 19.8 23.3 27.0 31.0
```

We may therefore determine the location of an extremum by determining the *roots* (arguments where the function value is zero) of the derivative function.

Since we are concerned only with real roots we will define a simple adverb for determining the value of a root in a specified interval, where the function values at the ends of the interval must differ in sign. The method used is sometimes called the *bisection* method; the interval is repeatedly halved in length by using the midpoint (that is, the mean) together with that endpoint for which the function value differs in sign. Thus:

```

m=. +/ % #
bis=. '2&{.0(m , ] #~ m ~:&(*@x.) ])' : 1
f y=. 1 4          Interval that bounds a root of f
1.75 _20
f bis y            One step of bisection method
2.5 1
f f bis y          Resulting interval still bounds a root
_3.35938 1.75
f bis^:0 1 2 3 4 y  Successive bisections
1      4
2.5    1
1.75   2.5
2.125  1.75
1.9375 2.125
f bis^:_ y          Limit of bisection
2 2
]root=. m f bis^:_ y Root is mean of final interval
2
f root
_3.55271e_14

```

A root of the derivative of *f* identifies an extremum of *f*:

```

f D z=. 0.5 1.5
2.125 _1.625
]droot=. m f D bis^:_ z
1.21718
f D droot
_1.78149e_13

```

When the derivative of *f* is increasing, the graph of *f* bends upward; when the derivative is decreasing, it bends downward. At a maximum (or minimum) point of the derivative, the graph of *f* therefore

changes its curvature, and the graph crosses its own tangent. Such a point is called a *point of inflection*.

Since an extremum of the derivative occurs at a zero of *its* derivative, an inflection point of *f* occurs at a zero of *f* D D . Thus:

```

fmt f D D x
3.8 2.7 1.7 0.7 0.3 1.1 1.9 2.7 3.4 4.0
4.6 5.1 5.5 5.9 6.3 6.5 6.7 6.9 7.0 7.0
7.0 6.9 6.7 6.5 6.2 5.9 5.5 5.1 4.6 4.0
3.4 2.7 1.9 1.1 0.2 0.7 1.7 2.7 3.8 5.0
6.2 7.5 8.9 10.3 11.8 13.3 14.9 16.5 18.2 20.0
21.8 23.7 25.7 27.7 29.8 31.9 34.1 36.3 38.6 41.0

```

```

* f D D 0 1
1 1
]infl=. m f D D bis^:_ (0 1)
0.472474

```

```

f D D infl
3.85681e_7

```

A graph of *f* will show that the curve crosses its tangent at the point *infl*.

C1 Test the assertion that *droot* is a local minimum of *f* .

```
[ f droot + _0.0001 0 0.0001
```

It is not a minimum, but a maximum.]

C2 What is the purpose of the phrase *2&{ . 0* in the definition of *bis*?

```
[ Remove the phrase and try f bis 1 3 ]
```

C3 For various coefficients *c*, make tables or graphs of the derivative *c&p* D to determine intervals bounding roots, and use them with *bis* to determine extrema of the polynomial *c&p*.

D. NEWTON'S METHOD

Although the bisection method is certain to converge to a root when applied to an interval for which the function at the endpoints differs in sign, this convergence is normally very slow. The derivative of the function can be used in a method that normally converges much faster, although convergence is assured only if the initial guess is "sufficiently near" the root.

The function $f = (j-1)*(j-2)$ has roots at 1 and 2. A rough sketch of the graph:

```
' *' { ~ f`0: GR 40 y=. 1x20*i.60
```

(using the utility GR from §1E) with a tangent drawn at the point $x, f x = .3$ intersecting the axis (the graph of 0:) at the point $d, 0$ shows that d is a much better approximation to the nearby root at 2 than is x . The length $x-d$ is clearly the *run* that produces the *rise* $f x$ with the *slope* $f D x$. Consequently, $d = x - (f x) / (f D x)$ is a better approximation to the root at x .

A root can be determined by repeated application of this process, using an adverb N as follows:

```
N=. (') - x. % x. D' : 1) (^:_ ) ("0)
5!:5 <'f'
0 1 2.5 _2 0.25 (+/% ([ * ] ^ i.@#[])"1 0)
f N 6
6.31662
f f N 6
_1.64799e_17
f N i. 7
0 _0.316625 2 2 2 6.31662 6.31662
f f N i. 7
0 0 0 0 0 2.55872e_17 _1.64799e_17
```

This use of the derivative to find a root is called *Newton's method*. Although it converges rapidly near a single root, the method may not converge to the root nearest the initial guess, and may not converge at all. The initial guess *droot* determined in the preceding section as a maximum point of f illustrates the matter; the derivative at the point is approximately 0, and division by it yields a very large value as the next guess:

```
f N droot
6.31662
```

Since the derivative of a polynomial function $c\&pic$ can be computed directly using the coefficients $\}.c*i.\#c$, it is possible to define a version of Newton's method that does not make explicit use of the derivative adverb. Thus:

```
dc=. ').'@[] * i.@#)@x. pic ]' :1 ("0)
NP=. ']' - x.&pic % x. dc' : 1 ("0) (^:_ )
```

```

c NP x
0 _0.316625 2 2 2 6.31662 6.31662
c&pic N x
0 _0.316625 2 2 2 6.31662 6.31662

```

The following utilities are convenient for experimenting with polynomials and their roots:

```

pir=. */@(--~"1 0)           Polynomial in terms of roots
(-1 1 1) pir x=. i.7
1 8 27 64 125 216 343

pic=. #.~&|. "1 0           Polynomial in terms of coeffs
1 3 3 1 pic x
1 8 27 64 125 216 343

pp=. +//. @(*/) "1          Polynomial product
1 2 1 pp 1 3 3 1
1 5 10 10 5 1

(1 2 1 pp 1 3 3 1) pic x
1 32 243 1024 3125 7776 16807
(1 2 1 pic x) * (1 3 3 1 pic x)
1 32 243 1024 3125 7776 16807

cfr=. pp/@(- ,. 1:) "1      Coefficients from roots
cfr -1 1 1
1 3 3 1

```

D1 Use Newton's method to determine the roots for which the bisection method was used in Section C.

E. KERNER'S METHOD

Kerner's method for the roots of a polynomial is a generalization of Newton's method; at each step it treats an n -element list as an approximation to the $<: \#c$ roots of the polynomial $c&pic$, and produces an "improved" approximation. We will first define and illustrate the use of an adverb κ such that $c \kappa b$ yields the $<: \#c$ (or $\#b$) roots of the polynomial with coefficients c :

```

k=. ']-x.&pic % (<0 1)&|:@((1&(*/\."1))@(-/~))' : 1
K=. k (^:_ )
b=. 1 2 3 4
jc=. cfr b+0.5               Coefficients of polynomial with
59.0625 _93 51.5 _12 1      roots at b+0.5

```

```

c k b
2.09375 2.46875 3.28125 4.15625

```

Single step of Kerner

```

c K b
1.5 2.5 3.5 4.5

```

Limit of Kerner
Roots of $c \&pic$

```

c k ^: (i.7) b
      1      2      3      4
2.09375 2.46875 3.28125 4.15625
1.20508 2.59209 3.7207 4.48213
1.45763 2.53321 3.50503 4.50413
1.49854 2.50154 3.49996 4.49997
      1.5      2.5      3.5      4.5
      1.5      2.5      3.5      4.5

```

Six steps of Kerner

```

]b=. 4?20

```

Random starting value

```

17 4 9 7

```

```

c K b
1.5 3.5 4.5 2.5

```

The adverb k applies only to a *normalized* coefficient c , that is, one whose last element (for the highest order term) is 1. Thus:

```

norm=. (] % {:) @ (+./\.@|@* # ])
norm 1 2 0 3 4 0 0
0.25 0.5 0 0.75 1

```

The polynomials $c \&pic$ and $(norm\ c) \&pic$ have the same roots, and $norm\ c$ is a suitable argument to the adverb k .

Kerner's method applies to polynomials with complex roots; however it will not converge to complex roots if the beginning guess is completely real. The following function `begin` provides a suitable beginning argument:

```

begin=. %:@-@i.@<:@#
begin 1 3 3 1
0 0j1 0j1.41421

```

For example, the coefficients $d=. cfr\ 1\ 2\ 2j3\ 4\ 2j\_4$ define a polynomial with two complex roots. Thus:

```

]roots=. (norm d) K begin d
4j8.33277e_19 2j3 2j_4 2j4.16767e_20 1j_4.60805e_20
clean=. (.@+.^: (=+) "0 Clean out negligible imaginary parts
sortm=. /:| Sort on magnitude
clean roots sortm clean roots
4 2j3 2j_4 2 1 1 2 2j3 4 2j_4

```

The definition of the adverb **k** (for a single step of Kerner) can be revised to give an alternative equivalent adverb by replacing the division (**÷**) by matrix division (**÷.**), and removing the phrase (**<0 1> &| : @**) that extracts the diagonal of the matrix produced by the subsequent phrase. Thus:

```
ak=. ']-x.&pic %. ((1&(*/\."1))@(-/~))' : 1
c ak b
2.09375 2.46875 3.28125 4.15625
```

In this form it is clear that the vector of *residuals* produced by **x.&pic** (the values of the function applied to the putative roots, which must all be reduced to zero) is divided by the matrix produced by the expression to the right of **%.** This expression produces the vector derivative with respect to each of the approximate roots; like the analogous case of the direct calculation of the derivative in the adverb **ND** it is a direct calculation of the derivative without explicit use of the vector derivative adverb **VD=.** ("1) (D. 1). These matters are left for exploration by the reader.

E1 Find all roots of the functions used in Section C.

E2 Define some polynomials that have complex roots, and use Kerner's method to find all roots.

F. DETERMINANT AND PERMANENT

The function **-/ . *** yields the *determinant* of a square matrix argument. For example:

```
det=. -/ . *
]m=. >3 1 4;2 7 8;5 1 6
3 1 4
2 7 8
5 1 6
det m
_2
```

The determinant is a function of rank 2 that produces a rank 0 result; its derivative is therefore a rank 2 function that produces a rank 2 result. For example:

```
MD=. ("2) (D. 1)
det MD m
34 28 _33
2 2 2
_20 _16 19
```

This result can be checked by examining the evaluation of the determinant as the alternating sum of the elements of any one column, each weighted by the determinant of its respective *complementary minor*, the matrix occupying the remaining rows and columns; the derivative with respect to any given element is its weighting factor. For example, the complementary minor of the leading element of **m** is the matrix **m00=** `>7 8;1 6`, whose determinant is **34**, agreeing with the leading element of the derivative.

Corresponding results can be obtained for the *permanent*, defined by the function `+/. *`. For example:

```
per=. +/. *          per m
                        350
per MD m
50 52 37
10 38 8
36 32 23
```

F1 Read the following sentences and try to state the meanings of the functions defined and the exact results they produce. Then enter the expressions (and any related expressions that you might find helpful) and again try to state their meanings and results.

```
m=. i. 4 4
less=. ([ (~ i.#@[ -.] )"2 0
m less 2
lessall=. less i.#"2
lessall m
minors=. <"2 lessall &.(|:"2) lessall m
```

[**less** yields its left argument with the item indexed by the right argument deleted. **lessall** yields an array obtained by successively deleting each item. The *complementary minor* of any element of a matrix is the matrix obtained by deleting the row and column in which the element lies. The result **minors** is the set of all complementary minors of the argument **m**]

F2 Enter and then comment upon the following sentences:

```
minors=. <"2 lessall &.(|:"2) lessall sq=. *:i. 4 4
det=. -/ . *
det@> minors
det D sq
(det D sq) % (det@> minors)
```

G. MATRIX INVERSE

The matrix inverse is a rank 2 function that produces a rank 2 result; its derivative is therefore a rank 2 function that produces a rank 4 result. For example:

```
<"2 (7.1) ": (mi v=. %. ) MD m
```

<u>289.0</u>	17.0	170.0	<u>238.0</u>	17.0	136.0	280.5
<u>238.0</u>	14.0	140.0	<u>196.0</u>	14.0	112.0	231.0
<u>280.5</u>	<u>16.5</u>	<u>165.0</u>	<u>231.0</u>	<u>16.5</u>	<u>132.0</u>	<u>272.2</u>
17.0	<u>1.0</u>	<u>10.0</u>	14.0	<u>1.0</u>	<u>8.0</u>	<u>16.5</u>
17.0	<u>1.0</u>	<u>10.0</u>	14.0	<u>1.0</u>	<u>8.0</u>	<u>16.5</u>
<u>17.0</u>	1.0	10.0	<u>14.0</u>	1.0	8.0	16.5
170.0	<u>10.0</u>	<u>100.0</u>	140.0	<u>10.0</u>	<u>80.0</u>	<u>165.0</u>
136.0	<u>8.0</u>	<u>80.0</u>	112.0	<u>8.0</u>	<u>64.0</u>	<u>132.0</u>
<u>161.5</u>	9.5	95.0	<u>133.0</u>	9.5	76.0	<u>156.8</u>

H. LINEAR FUNCTIONS AND OPERATORS

As discussed in §1E, a linear function distributes over addition, and any rank 1 linear function can be represented in the form $mp \& m^1$, where m is a matrix, and mp is the matrix product. For example:

```
r=. |."1
```

Rank 1 reversal

```
a=. 3 1 4 [ b=. 7 5 3
```

```
r a
```

```
r b
```

```
4 1 3
```

```
3 5 7
```

```
(r a)+(r b)
```

```
r(a+b)
```

Reversal is linear.

```
7 6 10
```

```
7 6 10
```

```
mp=. +/ . * "1 2
```

```
]m=. i. 3 4
```

```
0 1 2 3
```

```
4 5 6 7
```

```
8 9 10 11
```

```
L=. mp&m
```

A linear function

```
L a
```

```
L b
```

```
36 44 52 60
```

```
44 59 74 89
```

```
L a+b
```

```
80 103 126 149
```

```
VD=. ("1) (D. 1)
```

```
L VD a
```

```
0 1 2 3
```

```
4 5 6 7
```

```
8 9 10 11
```

```
=/~a
```

```
1 0 0
```

```
0 1 0
```

```
0 0 1
```

```
L =/~a
```

```
0 1 2 3
```

```
4 5 6 7
```

```
8 9 10 11
```

```
r VD a
```

```
0 0 1
```

```
0 1 0
```

```
1 0 0
```

```
r =/~a
```

```
0 0 1
```

```
0 1 0
```

```
1 0 0
```

```
per=. 2&A."1
```

```
per a
```

```
1 3 4
```

```
per VD a
```

```
0 1 0
```

```
1 0 0
```

```
0 0 1
```

The derivative of a linear function yields the matrix that represents it.

An identity matrix

A linear function applied to the identity matrix also yields the matrix that represents it.

The matrix that represents the linear function reversal

A permutation is linear.

A function such as $(\lambda 0 \ 1 \ 2) \cdot 0$ can be considered as a *family* of component functions. For example:

```
F=. (\lambda 0 1 2) \cdot 0
```

```
F 3
```

```
1 3 9
```

```
F y=. 3 4 5
```

```
1 3 9
```

```
1 4 16
```

```
1 5 25
```

The function $L @ F$ provides *weighted sums* or *linear combinations* of the members of the family F , and the adverb $L @$ is called a *linear operator*. Thus:

```
L @ F y
```

```
84 97 110 123
```

```
144 165 186 207
```

```
220 251 282 313
```

```
LO=. L @
```

```
F LO y
```

```
84 97 110 123
```

```
144 165 186 207
```

```
220 251 282 313
```

The linear function F applied to the results of the family of functions F

A linear operator

C=. 2&o.0(*&0 1 2)"0

Family of cosines (harmonics)

C y

```
1 0.989992 0.96017
1 0.653644 0.1455
1 0.283662 0.839072
```

C LO y

A Fourier series

```
3.72139 4.69157 5.66175 6.63193
3.77857 3.57772 3.37686 3.17601
5.57792 5.13333 4.68874 4.24415
```

The *tie* conjunction can be used to assemble two or more verbs into a *gerund*, a noun that has the force of a verb. A gerund can be *invoked* by the adverb `:0 to produce a single verb that is equivalent to the application of each of the verbs that comprise it, thereby producing an arbitrary family of functions. Thus:

g=. +`-`!

h=. g`:0"0

h x=. i. 4

```
0 0 1
1 1 1
2 2 2
3 3 6
```

h y

```
3 3 6
4 4 24
5 5 120
```

h LO y

```
36 42 48 54
176 200 224 248
940 1060 1180 1300
```

H1 Enter and experiment with the expressions of this section.

I. LINEAR DIFFERENTIAL EQUATIONS

If $f=. 2&o.$ and:

$F=. (f D. 0) \cdot (f D. 1) \cdot (f D. 2) \cdot :0 " 0$

$L=. mp\&c=. 1 0 1$

then F is a family of derivatives of f . If the function $L@F$ is identically zero, then the function f is said to be a solution of the *linear differential equation* defined by the linear function L . In the present example, f was chosen to be such a solution:

$L@F y=. 0.1*i.4$

```
0 0 0 0
```

The solution of such a differential equation is not necessarily unique; in the present instance the sine function is also a solution:

```
f=. 1&o.
L@F y=. 0.1*i.4
0 0 0 0
```

In general, the basic solutions of a linear differential equation defined by the linear function $L=. mp&c$ are $f=. ^@(*&sr)$, where sr is any one root of the polynomial $c&p$. In the present instance:

```
c
1 0 1
c K begin c      Roots of c&p using Kerner's method
0j_1 0j1
f=. s=. ^@(*&0j1)
L@F y
0 0 0 0
f=. t=. ^@(*&0j_1)
L@F y
0 0 0 0
```

Moreover, any linear combination of the basic solutions s and t is also a solution. In particular, the following are solutions:

```
u=. (s+t)%2"0      The cosine function 2&o.
v=. (s-t)%0j2"0      The sine function 1&o.
```

Since u is equivalent to the cosine function, this agrees with the solution f used at the outset.

II Enter the expressions of this section, and experiment with similar differential equations.

J. DIFFERENTIAL GEOMETRY

The differential geometry of curves and surfaces, as developed by Eisenhart in his book of that title [14], provides interpretations of the vector calculus that should prove understandable to anyone with an elementary knowledge of coordinate geometry. We will provide a glimpse of his development, beginning with a function similar to **PATH** of §1D, which Eisenhart calls a *circular helix*.

The following defines a circular helix in terms of an argument in degrees, with a rise of 4 units per revolution:

```

CH=. (1&o.0(%&180p_1) , 2&o.0(%&180p_1) , *4r360)"0
CH 0 1 90 180 360
      0          1          0
0.0174524    0.999848 0.0111111
      1 1.0842e_19          1
2.1684e_19          _1      2
4.33681e_19          1      4
D=. ("0) (D. 1)
CH D x
0.0174533    1.5233e_10 0.0111111
0.0174506    0.000304602 0.0111111
_1.37078e_8    0.0174533 0.0111111
_0.0174533    2.74156e_8 0.0111111
_0.0174533    5.48311e_8 0.0111111
CH D D x
1.32351e_10    0.00015233 _9.22207e_11
_5.3165e_6    0.000304502 2.75592e_10
_0.000304618    4.7737e_10 6.19929e_12
_9.56423e_10    0.000304618 3.09965e_12
_1.91369e_9    0.000304618 1.54982e_12

```

The derivatives produced by `CH D` in the expression above are the directions of the tangents to the helix; *their* derivatives produced by `CH D D` are the directions of the *binormals*. The binormal is perpendicular to the tangent, and indeed to the *osculating* (kissing) plane that touches the helix at the point given by `CH`.

These matters may be made more concrete by drawing the helix on a mailing tube or other circular cylinder. An accurate rendering of a helix can be made by drawing a sloping straight line on a sheet of paper and rolling it on the tube. A drawing to scale can be made by marking the point of overlap on the paper, unrolling it, and drawing the straight line with a rise of 4 units and a run of the length of the circumference. Finally, the use of a sheet of transparent plastic will make visible successive laps of the helix. Then proceed as follows:

1. Use a nail or knitting needle to approximate the tangent at one of the points where its directions have been computed, and compare with the computed results.
2. puncture the tube to hold the needle in the direction of the binormal, and again compare with the computed results.

3. Puncture a thin sheet of flat cardboard and hang it on the binormal needle to approximate the osculating plane.

4. Hold a third needle in the direction of the *principal* normal, which lies in the osculating plane perpendicular to the tangent.

To compute the directions of the principal normal we must determine a vector perpendicular to two other vectors. For this we can use the skew array used in §6I, or the following simpler *vector product* function:

```
vp=. (1&|.0[ * _1&|.0]) - (_1&|.0[ * 1&|.0])
a=. 1 2 3 { b=. 7 5 2
]q=. a vp b
_11 19 _9
a +/ . * q      b +/ . * q
0                0
```

As illustrated by the fact that we used an argument in radians for the helix defined by **PATH**, and degree arguments for the function **CH**, it is clear that the choice of the argument to describe a curve is rather arbitrary. As Eisenhart points out, it is possible to choose an argument that is intrinsic to the curve, namely the length along its path.

In the case of the helix defined by **CH**, it is easy to determine the relation between the path length and the degree argument. From the foregoing discussion of the paper tube model it is clear that the length of the helix corresponding to 360 degrees is the length of the hypotenuse of the triangle with sides 360 and 4. Consequently the definition of a function **df1** to give degrees from length is given by:

```
df1=. %&((%: +/ *: 4 360) % 360)
```

and the function **CH@df1** defines the helix in terms of its own length.

It is possible to modify the definition of the function **CH** to produce more complex curves, all of which can be modelled by a paper tube. For example:

1. Replace the constant multiple function for the last component by other functions, such as the square root, square, and exponential.
2. Multiply the functions for the first two elements by constants **a** and **b** respectively, to produce a helix on an elliptical

cylinder. This can be modelled by removing the cardboard core from the cylinder and flattening it somewhat to form an approximate ellipse.

K. AREAS AND VOLUMES

As remarked in §1B, the integral of a function may be interpreted as the area under its graph. To approximate integrals, we will use the adverb `∫` defined in §2C. For example:

```
(0&o.) ∫ 1  Approx area of quadrant of circle
0.785088
```

```
4 * (0&o.) ∫ 1  Approximation to pi
3.14035
```

```
*: ∫ x=. 1 2 3 4
0.333333 2.66667 9 21.3333
```

```
(^&3 % 3:) x
0.333333 2.66667 9 21.3333
```

The foregoing integral of the square function can be interpreted as the area under its graph. Alternatively, it can be interpreted as the *volume* of a three-dimensional solid as illustrated in Figure K1; that is, as the volume of a pyramid. In particular, the equivalent function `^&3 % 3:` is a well-known expression for the volume of a pyramid.

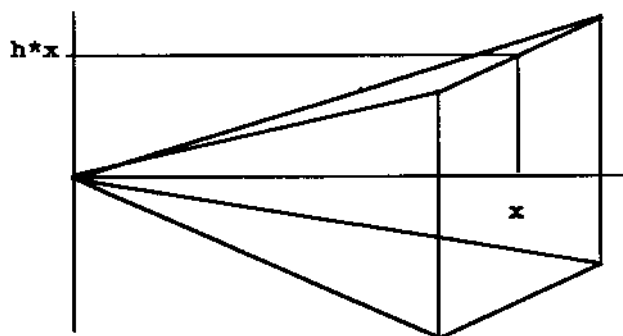


Figure K1

Similarly for a function that defines the area of a circle in terms of its radius:

```
ca=. 0.0*:0] " 0
ca x
3.14159 12.5664 28.2743 50.2655
ca I x
1.0472 8.37758 28.2743 67.0206
```

By drawing a figure analogous to Figure K1, it may be seen that the cone whose volume is determined by `ca I` can be generated by revolving the 45-degree line through the origin about the axis. The volume is therefore called a *volume of revolution*.

Functions other than `I` (the 45-degree line) can be used to generate volumes of revolution. For example:

```
cade=. ca0^0-
cade x
0.425168 0.0575403 0.00778723 0.00105389
cade I x
1.35821 1.54203 1.5669 1.57027
```

Area of circle whose radius
is the decaying exponential

Volume of revolution of
the decaying exponential

Because the expression `f I y` applies the function `f` to points ranging from 0 to `y`, the area approximated is the area over the same interval from 0 to `y`. The area under `f` from `a` to `b` can be determined as a simple difference. For example:

```
f=. ^3
f I b=. 4
64
(f I b) - (f I a)
60
```

`f I a=. 2`
4
-/f I b,a
60

However, this approach will not work for a function such as `%`, whose value at 0 is infinite. In such a case we may use the related function `%0(+&a)`, whose value at 0 is `%a`, and whose value at `b-a` is `%b`. Thus:

```
g=. %0(+&a)
g 0
0.5
g I b-a
0.693147
```

`g b-a`
0.25

The integral of the reciprocal from 2 to 4

`^. 2` The natural log of 2

0.693147

K1 Use integration to determine the areas and volumes of various geometrical figures, including cones and other volumes of revolution.

L. PHYSICAL EXPERIMENTS

Simple experiments, or mere observation of everyday phenomena, can provide a host of problems for which simple application of the calculus provides solutions and significant insights. The reason is that phenomena are commonly governed by simple relations between the functions that describe them, and their rates of change (that is, derivatives).

For example, the position of a body as a function of time is related to its first derivative (velocity), its second derivative (acceleration), and its third derivative (jerk). More specifically, if $p(t)$ gives the position at time t of a body suspended on a spring or rubber band, then the acceleration of the body ($p''(t)$) is proportional to the force exerted by the spring, which is itself a simple linear function of the position p .

If position is measured from the rest position (where the body rests after motion stops) this linear function is simply multiplication by a constant function c determined by the elasticity of the spring, and $c \cdot p$ must be equal and opposite to $m \cdot p''$, where the constant function m is the mass of the body. In other words, $(c \cdot p) - (m \cdot p'')$ must be zero.

This relation can be simplified to the form $0 = p - c^2 \cdot p''$, where c^2 is the constant function defined by $c^2 = m/c$. The function p is therefore (as seen in §I) the sine function, or, more generally, $p = (a \cdot \sin) + (b \cdot \cos)$, where a and b are constant functions.

This result is only an approximation, since a body oscillating in this manner will finally come to rest, unlike the sine and cosine functions which continue with undiminished amplitude. The difference is due to resistance (from friction with the air and internal friction in the rubber band) which is approximately proportional to the velocity. In other words, the differential equation:

$$0 = (d \cdot p) + (e \cdot p') + (f \cdot p'')$$

provides a more accurate relation.

As seen in §I, a solution of such a linear differential equation is given by e^{rt} , where r is a (usually complex) root of the polynomial $(d, e, f)p$. If $r = \alpha + j\omega$, then e^{rt} may also be written as $(e^{\alpha t}) * (e^{j\omega t})$, showing that the position function is a product of a decay function $(e^{\alpha t})$ and a periodic function $(e^{j\omega t})$ like the solution to the simpler case in which the (resistance) constant e was zero.

Because oscillations similar to those described above are such a familiar sight, most of us could perform the corresponding "thought experiment" and so avoid the effort of an actual experiment. However, the performance of actual experiments is salutary, because it commonly leads to the consideration of interesting related problems.

For example, direct observations of the effect of greater damping can result from immersing the suspended body in a pail of water. The use of a heavier fluid would increase the damping, and raise the following question: Could the body be completely damped, coming to rest with no oscillation whatever?

The answer is that no value of the decay factor α could completely mask the oscillatory factor $e^{j\omega t}$. However, a positive value of the factor f (the coefficient of p^2) will provide real roots r , resulting in non-oscillating solutions in terms of the hyperbolic functions $\sinh$ and $\cosh$. Such a positive factor cannot, of course, be realized in the experiment described.

The performance of actual experiments might also lead one to watch for other phenomena governed by differential equations of the same form. For example, if the function q describes the quantity of electrical charge in a capacitor whose terminals are connected through a resistor and a coil, then q is the *current* (whose value determines the voltage drop across the resistor), and $\dot{q}$ is *its* rate of change (which determines the voltage drop across the coil). In other words, the charge q satisfies the same form of differential equation that describes mechanical vibrations, and enjoys the same form of *electrical* oscillation.

Other systems concerning motion suggest themselves for actual or thought experiments:

- * The voltage generated by a coil rotating in a magnetic field.
- * The amount of water remaining in a can at a time t following the puncture of its bottom by a nail.

* The amount of electrical charge remaining in a capacitor draining through a resistor (used in circuits for introducing a time delay).

Coordinate geometry also provides problems amenable to the calculus. For example, $c = (1 \cos t, 2 \sin t)$ is a rank 1 0 function that gives the coordinates of a circle, and the gradient $c' = (-1 \sin t, 2 \cos t)$ gives the slope of its tangent. Similarly, $e = (a \cos t, b \sin t)$ gives the coordinates of an ellipse.

If we are indeed surrounded by phenomena so clearly and simply described by the calculus, why is it that so many students forced into calculus fail to see any point to the study? This is an important question, for which we will now essay some answers:

1. Emphasis on rigorous analysis of limits in an introductory course tends to obscure the many interesting aspects of the calculus which can be enjoyed and applied without it.

2. On the other hand, a superficial treatment that does not lead the student far enough to actually produce significant new results is likely to leave her uninterested. Textbook pictures of suspension bridges with encouraging but unhelpful remarks that calculus can be used to analyze the form assumed by the cables are more likely to discourage than stimulate a student.

3. The use of scalar notation makes it difficult to reach the interesting results of the vector calculus in an introductory course.

4. Although the brief treatments of mechanical and electrical vibrations given here may provide significant insights into their solutions, they would prove unsatisfactory in a text devoted to physics: they ignore the matter of relating the coefficients in the differential equations to the actual physical measurements (Does *mass* mean the same as *weight*? In what system of units are they expressed?); they ignore questions concerning the goodness of the approximation to the actual physical system; and they ignore the practicality of the computations required.

The treatment of such matters, although essential in a physics text, would make difficult its use by a student in some other discipline looking only for guidance in calculus.

Chapter 8: ANALYSIS

A. INTRODUCTION

To a math student conversant only with high-school algebra and trigonometry, the arguments used in Section 1B to determine the exact derivative of the cube (dividing the rise in the function value by the *run* x , and then setting x to zero in the resulting expression) might appear not only persuasive but conclusive. Moreover, the fact that the derivative so determined leads to consistent and powerful results would only tend to confirm a faith in the validity of the arguments.

On the other hand, a more mature student familiar with the use of rigorous axiomatic and deductive methods would, like Newton's colleagues at the time of his development of what came to be the calculus, have serious qualms about the validity of assuming a quantity x to be non-zero and then, at a convenient point in the argument, asserting it to be zero.

Should a student interested primarily in the practical results of the calculus dismiss such qualms as pedantic "logic-chopping", or are there important lessons to be learned from the centuries-long effort to put the calculus on a "firm" foundation? If so, what are they, and how may they be approached?

The important lesson is to appreciate the limitations of the methods employed, and to learn the techniques for assuring that they are being properly observed. As Morris Kline says in the preface to his *Mathematics: The Loss of Certainty* [15]:

But intellectually oriented people must be fully aware of the powers of the tools at their disposal. Recognition of the limitations, as well as the capabilities, of reason is far more beneficial than blind trust, which can lead to false ideologies and even to destruction.

Concerning "This history of the illogical development [of the calculus] ...", Kline states (page 167):

But there is a deeper reason. A subtle change in the nature of mathematics had been unconsciously made by the masters. Up to about 1500, the concepts of mathematics were immediate realizations of or abstractions from experience. ... In other

words, mathematicians were [now] contributing concepts rather than abstracting ideas from the real world.

Chapter VII of Kline provides a brief and readable overview of ingenious attempts to put the calculus on a firm basis, and equally ingenious refutations. Students are urged to read it in full, and perhaps to supplement it with Lakatos' equally readable account of the interplay between proof and refutation in mathematics. In particular, a student should be aware of the fact that weird and difficult functions sometimes brought into presentations of the calculus are included primarily because of their historical role as refutations. The words of Poincare (quoted by Kline on page 194) are worth remembering:

When earlier, new functions were introduced, the purpose was to apply them. Today, on the contrary, one constructs functions to contradict the conclusions of our predecessors and one will never be able to apply them for any other purpose.

The central concept required to analyze derivatives is the *limit*; it is introduced in Section B, and applied to *series* in Section D.

B. LIMITS

The function `h=. (*:-9:) % (]-3:)` applied to the argument `a=. 3` yields the meaningless result of zero divided by zero. On the other hand, a list of arguments that differ from `a` by successively smaller amounts appear to be approaching the limiting value `g=. 6`. Thus:

```
g=. 6:
h=. (*:-9:) % (]-3:)
a=. 3
h a
0
|i=. , (+,-)"0 (10^-i.5)
1 _1 0.1 _0.1 0.01 _0.01 0.001 _0.001 0.0001 _0.0001
a+i
4 2 3.1 2.9 3.01 2.99 3.001 2.999 3.0001 2.9999
h a+i
7 5 6.1 5.9 6.01 5.99 6.001 5.999 6.0001 5.9999
|(g-h) a+i
1 1 0.1 0.1 0.01 0.01 0.001 0.001 0.0001 0.0001
```

We might therefore say that $h(x)$ approaches a limiting value, or *limit*, as x approaches a , even though it differs from $h(a)$. In this case the limit is the constant function $g(x) = 6$.

We make a more precise definition of limit as follows: The function h has the limit g at a if there is an associated *frame* function $f(x)$ such that for any positive value of ϵ , the expression $\epsilon > |g(h(y)) - g(a)|$ is true for any y such that $(|y-a| < f(x))$. In other words, for any positive value ϵ , however small, there is a value $d = f(x)$ such that $h(y)$ differs from $g(y)$ by no more than ϵ , provided that y differs from a by no more than d .

Figure B provides a graphic picture of the role of the frame function: $d = f(x)$ specifies the half-width of a frame such that the horizontal boundary lines at ϵ and $-\epsilon$ are not crossed by the graph of $g \circ h$ within the frame.

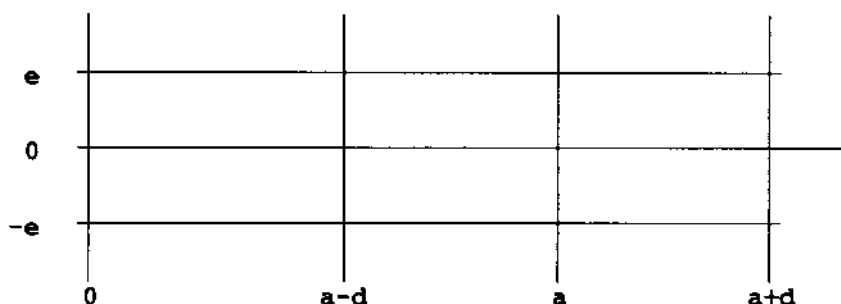


Figure B

As illustrated at the beginning of this section, the function $g(x) = 6$ is the apparent limit of the function $h(x) = (x-3)^2$ at the point $a = 3$. The simple frame function $f(x) = |x-3|$ suffices, as illustrated (and later proved) below:

```

fx = |x - 3|
a = 3
e = 0.2
d = a - fx
0.2
ji = , (+, -) "0.5%~>:1.5
0.2 _0.2 0.4 _0.4 0.6 _0.6 0.8 _0.8 1 _1

```

```

]j=. d*i
0.04 _0.04 0.08 _0.08 0.12 _0.12 0.16 _0.16 0.2 _0.2
| (g-h) a+j
0.04 0.04 0.08 0.08 0.12 0.12 0.16 0.16 0.2 0.2
e>: | (g-h) a+j
1 1 1 1 1 1 1 1 1 1

```

We now offer a proof that $\epsilon x = .1$ suffices, by examining the difference function $g-h$ in a series of simple algebraic steps as follows:

```

g-h
6: - (*:-9:) % (1-3:)      Definitions of g and h
6: + (*:-9:) % (3:-1)
((6:*3:-1)+( *-9:)) % (3:-1)
((18"0-6:*1)+( *-9:)) % (3:-1)
((9:-6:*1)+*:) % (3:-1)
((3:-1)* (3:-1)) % (3:-1)
3:-1

```

Cancel terms, but domain now excludes 3

To recapitulate: for the limit point $a = .3$ we require a frame function ϵx such that the magnitude of the difference $(g-h)$ at the point $a+a \epsilon x$ shall not exceed ϵ . We have just shown that the difference function $(g-h)$ is equivalent to $(3:-1)$. Hence:

```

| (g-h) a + a \epsilon x \epsilon
| (3:-1) 3+3 \epsilon x \epsilon      Definition of (g-h) and limit point
| 3-(3+3 \epsilon x \epsilon)
| -3 \epsilon x \epsilon
| 3 \epsilon x \epsilon

```

Consequently, the simple function $\epsilon x = .1$ will suffice.

In the preceding example, the limiting function was a constant. We will now examine a more general case of the limit of the secant slope (that is, the derivative) of the fourth-power function. Thus:

```

f=. ^4
h=. [ %~ ] -&f --
x=. 0 1 2 3 4
]a=. 10^->:i. 6
0.1 0.01 0.001 0.0001 1e_5 1e_6

```

```

a h"0/ x
_0.001 3.439 29.679 102.719 246.559
_1e_6 3.9404 31.7608 107.461 255.042
_1e_9 3.994 31.976 107.946 255.904
_1e_12 3.9994 31.9976 107.995 255.99
_1e_15 3.99994 31.9998 107.999 255.999
_1e_18 3.99999 32 108 256

```

The probable limiting function as the run `a` approaches zero could be obtained by using the function `FIT` of §1E to determine the approximate polynomial, but the last row of the foregoing result suggests the function `4:*^3`. Thus:

```

g=. 4:*^3
g x
0 4 32 108 256
a=. 1e_6
(g-a&h) x
1e_18 6e_6 2.4e_5 5.4e_5 9.6e_5

```

In simplifying the expression for the difference $(g-a&h) \ x$ we will use functions for the polynomial and for weighted binomial coefficients as illustrated below:

```

p=. #.~&|. "1 0
w=. (]^i.0-0>:0[]) * i.0>:0[ ! [
x=. 0 1 2 3 4 5
a=. 0.1
(x-a)^4
0.0001 0.6561 13.0321 70.7281 231.344 576.48
(4 w -a) p x
0.0001 0.6561 13.0321 70.7281 231.344 576.48
4 w -a
0.0001 _0.004 0.06 _0.4 1

```

The following expressions for the difference can each be entered so that their results may be compared:

```

(g-a&h) x
(4*x^3)-a %~ (f x) - (f x-a)
(4*x^3)-a %~ (x^4) - (x-a)^4
(0 0 0 4 0 p x) - a %~ (0 0 0 0 1 p x) - (4 w -a) p x
a %~ ((a*0 0 0 4 0) p x) - (0 0 0 0 1 p x) - (4 w -a) p x
a %~ (1 _4 6 * a^ 4 3 2) p x
(1 _4 6 * a^3 2 1) p x

```

We will now obtain a simple upper bound for the magnitude of the difference (that is, $|g-a\&h|$ x), beginning with the final expression above, and continuing with a sequence of expressions that are greater than or equal to it: (If the expressions are to be entered, x should be set to a scalar value, as in $x=.5$, to avoid length problems)

(1 _4 6 * a^3 2 1) p x	Magnitude of (g-a&h) x
+/1 _4 6*(a^3 2 1)*x^i.3	Polynomial as sum of terms
+/(1 _4 6)*(a^3 2 1)*(x^i.3)	Sum of mags>:mag of sum
+/1 4 6*(a^3 2 1)* x^0 1 2	a is non-negative
+/6*(a^3 2 1)* x^0 1 2	
+/6*a* x^0 1 2	For a<1, the largest term is a^1
6*a+ x^0 1 2	
a* (6+ x^0 1 2)	

The final expression provides the basis for a frame function: if $a=.e \& (6+|x^0 1 2)$, then the magnitude of the difference $|g-a\&h|$ x will not exceed a . For example:

```
e=. 0.001
a=. e & (6+|x^0 1 2)
| (g-a&h) x
0.000806451
```

C. CONTINUITY

Informally we say that a function f is continuous in an interval if its graph over the interval can be drawn without lifting the pen. Formally, we define a function f to be continuous in an interval if it possesses a limit at every point in the interval.

For example, the integer part function $<.$ is continuous in the interval from 0.1 to 0.9, but not in an interval that contains integers.

D. CONVERGENCE OF SERIES

In §2E we introduced the exponential coefficients function $ec=.e\&!$, used it to generate coefficients for a polynomial that approximates its own derivative, and defined the growth function (exponential) as the limiting value for an infinite number of terms. Since the coefficients

produced by `ec` decrease rapidly in magnitude (the 20th element is `%!19`, approximately `8e_18`), it seemed reasonable to assume that the polynomial `(ec i.n)*p` would converge to a limit for large `n` even when applied to large arguments. We will now examine more carefully the conditions under which a sum of such a series approaches a limit.

It might seem that the sum of a series whose successive terms approach zero would necessarily approach a limiting value. However, the series `%e>:0i . n` provides a counter example, since (by considering sums over successive groups of `2^i . k` elements) it is easy to show that its sum can be made as large as desired.

If at a given term `t` in a series the remaining terms are decreasing in such a manner that the magnitudes of the ratios between each pair of successive terms are all less than some value `x` less than 1, then the magnitude of the sum of the terms after `t` is less than the magnitude of `t*(1-x)`; if this quantity can be shown to approach 0, the sum of the entire series therefore approaches a limit.

This can be illustrated by the series `x^i.n`, which has a fixed ratio `x`, and has a sum equal to `(1-x^n) / (1-x)`. For example:

```
S=. [ ^ i.@]
T=. (1:-^)% (1:-[])
r=. 3
n=. 10

r S n
1 3 9 27 81 243 729 2187 6561 19683

+ / r S n
29524

r T n
29524
```

A proof of the equivalence of `T` and the sum over `S` can be based on the patterns observed in the following:

```
(1,-r) */ r S n
1 3 9 27 81 243 729 2187 6561 19683
_3 _9 _27 _81 _243 _729 _2187 _6561 _19683 _59049
]dsums=.+ / . (1,-r) */ r S n
1 0 0 0 0 0 0 0 0 0 _59049

-r^10 + / dsums (1-r) * r T n
_59049 _59048 _59048
```

If the magnitude of x is less than 1, the value of x^n in the numerator of $x^n/n!$ approaches zero for large n , and the numerator itself therefore approaches 1; consequently, the result of $x^n/n!$ approaches $x/(1-x)$ for large n .

The expression $e^{x-j} - 1$ gives a pair of successive coefficients of the polynomial approximation to the exponential, and $x/(e^{x-j} - 1)$ gives their ratio. For example:

```
j=. 4
ec j-0 1
0.0416667 0.166667
%/ec j-0 1      %j
0.25            0.25
```

The ratio of the corresponding terms of the polynomial $(e^{x-j} - 1)/j$ applied to x is x times this, namely, x^j/j . At some point this ratio becomes less than 1, and the series for the exponential therefore converges. Similar proofs of convergence can be made for the series for the circular and hyperbolic sines and cosines, after removing the alternate zero coefficients.

Another generally useful proof of convergence can be made for certain series by establishing upper and lower bounds for the series. This method applies if the elements alternate in sign and decrease in magnitude.

We will illustrate this by first developing a series approximation to the arctangent, that is, the inverse tangent $\arctan$. The development proceeds in the following steps:

1. Derivative of the tangent
2. Derivative of the inverse tangent
3. Express the derivative as a polynomial in the tangent
4. Express the derivative as the limit of a polynomial
5. Integrate the polynomial
6. Apply the polynomial to the argument 1 to get a series whose sum approximates the arctangent of 1 (that is, one-quarter π):

```
1x=. 0,1x6p1,1x4p1,1x3p1
0 0.523599 0.785398 1.0472
sin=.1&o. [. cos=.2&o. [. tan=.3&o. [. arctan=._3&o.
sin x
0 0.5 0.707107 0.866025
```

```

cos x
1 0.866025 0.707107 0.5
tan x
0 0.57735 1 1.73205
(sin % cos) x
0 0.57735 1 1.73205
INV=. ^:_1
tan INV tan x
0 0.523599 0.785398 1.0472
D=. ("0) (D. 1)

tan D
(sin % cos) D
(sin % cos) * (sin D % sin) - (cos D % cos)
tan * (cos % sin) - (-@sin % cos)
tan * %@tan +tan
1: + tan * tan
1: + *:@tan
tan INV D
1: % tan D @ (tan INV)
1: % (1: + *:@tan) @ (tan INV)
1: % (1:@(tan INV)) + *:@tan@(tan INV)
1: % 1: + *:@
1: % 1: + *:
%@(1:+*: )
p=. #.~&|. "1 0
c=. 1 0 1
% c&p x
b=. 1 0 _1 0 1 0 _1 0 1 0 _1
c */ b
+//. c */ b

```

Definition of tan

07§2B

§2I

Derivative of tangent (End of proof)

07§2B

Derivative of inverse tangent (End of proof)

Derivative of inverse tan as reciprocal of a polynomial

Coeffs of approx reciprocal

Product polynomial shows that b&p is approximate reciprocal of c&p

```

%@(1:+:) x
1 0.784833 0.618486 0.476958
    b&p x                      Better approx needs more terms of b
1 0.7845 0.584414 _0.352555
    int=. 0: , ] % 1: + i.&#
    a=. int b                  The function a&p is the integral of b&p
    a&p x                      Approximation to arctangent
0 0.482334 0.6636 0.736276
    tan INV x
0 0.482348 0.665774 0.808449
    7.3 ": 8{. a              The coeffs for arctan are reciprocals of odds
0.000 1.000 0.000 _0.333 0.000 0.200 0.000 _0.143
    1r4p1 , a p 1              Arctan 1 is one-quarter pi
0.785398 0.744012
    +/a                        Polynomial on 1 is sum of coefficients
0.744012
    gaor=. _1&^@i. * 1: % 1: + 2: * i.
    gaor 6                      Generate alternating odd reciprocals
1 _0.333333 0.2 _0.142857 0.111111 _0.0909091
    +/\gaor 6
1 0.666667 0.866667 0.72381 0.834921 0.744012
    7 2 $ +/\ gaor 14
    1 0.666667
0.866667 0.72381
0.834921 0.744012
0.820935 0.754268
0.813091 0.76046
0.808079 0.764601
0.804601 0.767564
    1r4p1 , +/\gaor 1000
0.785398 0.785148

```

First column (sums of odd number of terms) are decreasing upper bounds of limit. Second column (sums of even number of terms) are increasing lower bounds of limit.

D1 Test the derivations in this section by enclosing a sentence in parentheses and applying it to an argument, as in `(1: + *: @tan) x`

D2 Prove that a decreasing alternating series can be bounded as illustrated. [Group pairs of successive elements to form a sum of positive or negative terms]

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=	Self-Classify • Equal	Is (Local)	Is (Global)
<	Box • Less Than	Floor • Lesser of	Decrem • Less Or Eq
>	Open • Larger Than	Ceiling • Larger of	Increm • Larg Or Eq
—	Negative Sign //Infinity	<i>Indeterminate</i>	Infinity
+	Conjugate • Plus	Real / Imag • GCD (Or)	Double • Not-Or
*	Signum • Times	Polar • LCM (And)	Square • Not-And
-	Negate • Minus	Not (1-) • Less	Halve • Match
%	Reciprocal • Divide	Matrix Inv • Mat Divide	Square Root • Root
^	Exponential • Power	Natural Log • Log	Power • Chain
\$	Shape Of • Shape	Suite	Self-Reference
~	<i>Reflex • Pass • EVOKE</i>	Nub •	Nub Sieve • Not-Eq
	Magnitude • Residue	Reverse • Rotate (Shift)	Transpose
.	Det • Dot Product	Even	Odd
:	Explicit Definition	Obverse	Adverse
,	Ravel • Append Items	Ravel Items • Append	Itemize • Laminate
;	Raze • Link	Cut	Word Formation •
#	Tally • Copy	Base 2 • Base	Antibase 2 • Antib
!	Factorial • Out Of	Fit (Customize)	Foreign
/	<i>Insert • Table • INSERT</i>	<i>Oblique • Key</i>	Grade Up • Sort
\	<i>Prefix • Infix • TRAIN</i>	<i>Suffix • Outfix</i>	Grade Down • Sort
[	Same • Left	Lev	
]	Same • Right	Dex	
{	Catalogue • From	Head • Take	Tail •
}	<i>Amend</i>	Behead • Drop	Curtail •
"	Rank • CONSTANT	Do • Do left if error	Format
`	Tie (Gerund)		Evoke Gerund
@	Atop	Agenda	At
&	Bond /Compose	Under (Dual)	Appose
?	Roll • Deal		
)	Label	a . Alphabet	A . Atomic Permute
b .	Boolean	c . Characteristic	C . Cyc-Dir • Perm
D .	Derivative	e . Raze In • Member (In)	E . • Member of Int
f .	<i>Fix Verb</i>	i . Integers • Index Of	j . Imagin • Cmplx
NB .	Comment	o . Pi Times • Circle	p . Polynomial
r .	Angle • Cmplx	x . Left Argument	y . Right Argument
0 :	Zero 1 : One (to 9 :)		

Appendix A: Language Summary

